Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 245–252.

Two remarkable doubly exponential series transformations of Ramanujan

BRUCE C BERNDT and JAMES LEE HAFNER*

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA

*IBM Research Division, Almaden Research Center, K53/802, 650 Harry Road, San José, CA 95120-6099, USA

Dedicated to the memory of Professor K G Ramanathan

Abstract. The purpose of this note is to prove two doubly exponential series transformations found in Ramanujan's second notebook.

Keywords. Doubly exponential series; Bernoulli numbers; gamma function; Poisson summation.

Ramanujan's notebooks [2] contain many fascinating theorems, which, it would seem, would never have been discovered by any other person. As excellent illustrations of this, Ramanujan offers transformation formulas for two doubly exponential series. It is very surprising that such elegant transformations exist. Although beautiful by themselves, we think that they will be useful in other investigations. These two remarkable series identities are stated without proof by Ramanujan on page 279 in his second notebook [2] and are numbered 4) and 5) on that page. The purpose of this note is to provide the missing proofs.

Ramanujan's two formulas can be stated as follows. First, if $\alpha\beta = 2\pi$, then

$$\alpha \sum_{k=0}^{\infty} \exp(-ne^{k\alpha}) = \alpha\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} n^k}{k!(e^{k\alpha}-1)}\right\} - \gamma - \log n + 2\sum_{k=1}^{\infty} \varphi(k\beta), \quad (1)$$

where

$$\varphi(\beta) = \sqrt{\frac{\pi}{\beta\sinh(\pi\beta)}}\cos\left(\beta\log\frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1\cdot 2\beta} + \dots\right). \quad (2)$$

Second, if $\alpha\beta = \pi/2$ then

$$\alpha \sum_{k=0}^{\infty} (-1)^k \exp(-ne^{(2k+1)\alpha}) = \alpha\left\{\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k n^k}{k!(e^{k\alpha}+e^{-k\alpha})}\right\} + \sum_{k=0}^{\infty} (-1)^k \psi((2k+1)\beta), \quad (3)$$

where

$$\psi(\beta) = \sqrt{\frac{\pi}{\beta\sinh(\pi\beta)}}\sin\left(\beta\log\frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1\cdot 2\beta} + \frac{B_4}{3\cdot 4\beta^3} - \dots\right). \quad (4)$$