Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 237–243.

Gaussian quadrature in Ramanujan's Second Notebook

RICHARD ASKEY

Department of Mathematics, Van Vleck Hall, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

Dedicated to the memory of Professor K G Ramanathan

Abstract. Ramanujan's notebooks contain many approximations, usually without explanations. Some of his approximations to series are explained as quadrature formulas, usually of Gaussian type.

Keywords. Gaussian quadrature; series approximations; Ramanujan.

1. Introduction

K G Ramanathan was a gentle man who had a strong sense of duty. Part of his duty was the understanding of Ramanujan and his mathematics, and we can all feel pleased that he helped us understand some of the mathematics Ramanujan did. In light of his work on Ramanujan's work on modular functions, continued fractions, and hypergeometric and basic hypergeometric functions, it is appropriate to dedicate a paper to his memory which deals with material from Ramanujan's Notebooks. The particular questions below deal with orthogonal polynomials, although it is very unlikely Ramanujan knew this. He was just looking for nice approximations he could compute easily, and attractive explicit formulas.

Ramanujan's approximations to certain series which I can explain are:

$$\varphi(0)+\frac{x}{1!}\varphi(1)+\frac{x^2}{2!}\varphi(2)+\frac{x^3}{3!}\varphi(3)+\dots \tag{1.1}$$

$$= e^x\varphi(x) \text{ as the first approximation,} \tag{1.1a}$$

$$= e^x\left\{\frac{\sqrt{1+4x}-1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1+\sqrt{1+4x}}{2}\right)\right.$$
$$\left.+\frac{\sqrt{1+4x}+1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1-\sqrt{1+4x}}{2}\right)\right\} \tag{1.1b}$$

$$= e^x\left\{\frac{2}{3}\varphi(x)+\frac{\sqrt{1+12x}-1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1+\sqrt{1+12x}}{2}\right)\right.$$
$$\left.+\frac{\sqrt{1+12x}+1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1-\sqrt{1+12x}}{2}\right)\right\}. \tag{1.1c}$$