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Modular equations and Ramanujan's Chapter 16, Entry 29

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Dedicated to the memory of my friend, Professor K G Ramanathan

Abstract. In this paper we illustrate how some of the classical modular equations can be proved by using only Ramanujan's summation (see (1.1)) and dispensing completely with the Schröter-type methods.

Keywords. Modular equations; Rogers-Ramanujan functions.

1. Introduction

I first met K G Ramanathan because of our mutual interest in Ramanujan. He came to Penn State in February 1982 at my invitation to give our colloquium. I had been through a particularly trying week and was exhausted to say the least. Ramanathan presented a beautiful lecture explaining and extending work from Ramanujan's Lost Notebook [9]–[13]. I remember few mathematics talks as fondly as I remember that one. The beauty of the work truly revived my spirits.

General interest in Ramanujan's work has been intense in recent years due in no small part to the magnificient edited versions of Ramanujan's Notebooks [2], [3], [4] carefully prepared by Bruce Berndt.

This paper will be devoted to further considerations of modular equations, especially those of degrees 3 and 5. Berndt [4; pp. 6–7] and Hardy [7; Ch. 12] discuss several approaches to modular equations. Succinctly stated they are: (1) the Legendre-Jacobi method using differential equations for elliptic functions [7; §§ 12·4–12·7]; (2) Schröter's method requiring ingenious rearrangements of double theta series [4; p. 73]; (3) the theory of modular forms [4; p. 7], and (4) Ramanujan's method.

Both Hardy and Berndt are uncertain about Ramanujan's method for the excellent reason that he never revealed it. He merely stated his discoveries without proof, and as Berndt puts it "... found more modular equations than all of his predecessors put together."

To prove Ramanujan's formulas both Hardy [7; Ch. 12] and Berndt [4] mix Schröter's method, algebraic manipulation of series and products (what Hardy [7; pp. 220–221] calls "trivial" relations), and Ramanujan's ${}_1\psi_1$-summation [4; p. 32, Entry 17] rewritten as

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n t^n}{(b)_n} = \frac{(b/a, at, q/(at), q; q)_\infty}{(q/a, b/(at), b, t; q)_\infty}, \tag{1.1}$$