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Reduction theory over global fields

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Dedicated to the memory of Professor K G Ramanathan

Abstract. The paper contains an exposition of the basic results on reduction theory in reductive groups over global fields, in the adelic language. The treatment is uniform: number fields and function fields are on an equal footing.

Keywords. Reduction theory; global fields; number fields; function fields.

Introduction

The basic results on reduction theory for a linear algebraic group G over the field of rational numbers were established by Borel and Harish–Chandra in [3]. One of these results is the construction of a fundamental set for an arithmetic subgroup Γ of the real Lie group $G(\mathbf{R})$. For another one, the criterion for compactness of the quotient $G(\mathbf{R})/\Gamma$, a more direct method of proof was given by Mostow and Tamagawa [8]. Godement and Weil [5] showed that this method can also be used to obtain fundamental sets. They used the language of adeles.

Reduction theory for linear algebraic groups over number fields is reduced to groups over $\mathbf{Q}$ by restriction of the ground field. For groups over global fields of positive characteristic, i.e. function fields of dimension one over a finite field of constants, the method of Mostow and Tamagawa can also be used, but only under some restrictions on the characteristic (see [1]). Using another method, involving the study of semi-simple group schemes over complete curves, Harder [6] proved the basic results over function fields without restrictions on the characteristic.

Some 25 years ago, in unpublished seminar notes, I tried to give a uniform treatment of the reduction theory over global fields, by the method of [5], also using Harder's idea to employ Galois descent. This attempt was not successful; there was a gap in the notes. However, they contain a proof of the compactness theorem.

In the meantime, no uniform treatment of the basic results on reduction theory seems to have appeared in the literature. The present note, which is to a large extent expository, attempts to give such a treatment. The method is essentially that of the old notes. But I have abandoned the method of Mostow and Tamagawa altogether. Galois descent is used instead.

In applying the method of [8] (and its extension in [5]) one encounters a somewhat subtle question. This method seems to involve an application of the following strong version of the Hilbert–Mumford theorem. Let G be a reductive group over a field k, acting linearly in a vector space V, everything being defined over k. Let $\xi \in V(k)$ be a