

Finite arithmetic subgroups of GL_n , III

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Let G be an algebraic group in $GL_n(\mathbb{C})$ defined over \mathbb{Q} , and K an algebraic number field with the maximal order O_K . If the group $G(O_K)$ of rational points of G in $M_n(O_K)$ is a finite group and if it satisfies a certain condition, which is satisfied, for example, when K is a nilpotent extension of \mathbb{Q} and 2 is unramified, then $G(O_K)$ is generated by roots of unity in K and $G(\mathbb{Z})$.

Keywords. Algebraic group; algebraic number field; quadratic form; finite arithmetic subgroup.

Let K be a Galois extension of the rational number field \mathbb{Q} with Galois group $G(K/\mathbb{Q})$, and O_K the maximal order of K . Let G be a finite subgroup in $GL_n(O_K)$ such that G is stable under the action of $G(K/\mathbb{Q})$, that is $u(g) := (u(g_{ij})) \in G$ for every $g := (g_{ij}) \in G$ and $u \in G(K/\mathbb{Q})$. Our problem is whether G is of A -type in the sense of [2], that is, whether there is an element $h \in GL_n(\mathbb{Z})$ such that

$$\{hgh^{-1} | g \in G\} \subset \{\text{diag}(\varepsilon_1 A_1, \dots, \varepsilon_m A_m) | A_i \in GL_{n_i}(\mathbb{Z}), \varepsilon_i: \text{root of unity}\}.$$

Here $\text{diag}(\varepsilon_1 A_1, \dots, \varepsilon_m A_m)$ denotes the matrix in which $\varepsilon_1 A_1, \dots, \varepsilon_m A_m$ are diagonally arranged. If, hence $\{\pm 1\}$ are the only roots of unity in K , then G being of A -type means $G \subset GL_n(\mathbb{Z})$.

The lattice-theoretic meaning is: Let L be a free module of rank n over \mathbb{Z} , and let \mathcal{G} be a linear algebraic group in GL_n defined over \mathbb{Q} . Suppose that K is a Galois extension of \mathbb{Q} and that $\mathcal{G}(O_K)$ is a finite group, which is canonically regarded as a subgroup of automorphisms of $O_K L$. Then $\mathcal{G}(O_K)$ being of A -type implies that there exists a direct sum decomposition $L = \bigoplus_{i=1}^k L_i$ so that every $\sigma \in \mathcal{G}(O_K)$ and for roots of unity $\varepsilon_i \in K$ dependent on σ , we have

$$\varepsilon_i \sigma(L_i) = L_i \text{ for } i = 1, \dots, k.$$

We know that the above question is affirmative if either K is totally real and $G(K/\mathbb{Q})$ is nilpotent or $G(K/\mathbb{Q})$ is abelian ([1], [2]). The aim of this paper is to show that this is affirmative if $G(K/\mathbb{Q})$ is nilpotent and the complex conjugation induces an element of the center of $G(K/\mathbb{Q})$. In §2, we give miscellaneous results.

1. Main result

Lemma 1. Let K/\mathbb{Q} be a Galois extension with Galois group $\Gamma := G(K/\mathbb{Q})$. Denote by W the set of all roots of unity in K . We fix an element $a_\sigma \in W$ for $\sigma \in \Gamma$. If $a_{\mu\sigma} = a_\mu \mu(a_\sigma)$