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On the zeros of a class of generalised Dirichlet series–XIV

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Dedicated to the memory of Professor K G Ramanathan

Abstract. We prove a general theorem on the zeros of a class of generalised Dirichlet series. We quote the following results as samples.

Theorem A. *Let* $0<\theta<\frac{1}{2}$ *and let* $\{a_n\}$ *be a sequence of complex numbers satisfying the inequality* $\left|\sum_{m=1}^{N} a_m - N\right| \leqslant (\frac{1}{2}-\theta)^{-1}$ *for* $N=1,2,3,\ldots$, *also for* $n=1,2,3,\ldots$ *let* α_n *be real and* $|\alpha_n| \leqslant C(\theta)$ *where* $C(\theta)>0$ *is a certain (small) constant depending only on* θ. *Then the number of zeros of the function*

$$\sum_{n=1}^{N} a_n(n+\alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n(n+\alpha_n)^{-s} - n^{-s})$$

in the rectangle $(\frac{1}{2}-\delta \leqslant \sigma \leqslant \frac{1}{2}+\delta,\ T \leqslant t \leqslant 2T)$ *(where* $0<\delta<\frac{1}{2}$*) is* $\geqslant C(\theta,\delta)\, T \log T$ *where* $C(\theta,\delta)$ *is a positive constant independent of* T *provided* $T \geqslant T_0(\theta,\delta)$ *a large positive constant.*

Theorem B. *In the above theorem we can relax the condition on* a_n *to* $\left|\sum_{m=1}^{N} a_m - N\right| \leqslant (\frac{1}{2}-\theta)^{-1} N^{\theta}$ *and* $|a_N| \leqslant (\frac{1}{2}-\theta)^{-1}$. *Then the lower bound for the number of zeros in* $(\sigma \geqslant \frac{1}{2}-\delta,\ T \leqslant t \leqslant 2T)$ *is* $> C(\theta,\delta)\, T \log T (\log\log T)^{-1}$. *The upper bound for the number of zeros in* $(\sigma \geqslant \frac{1}{2}+\delta,$ $T \leqslant t \leqslant 2T)$ is $O(T)$ *provided* $\sum_{n \leqslant x} a_n = x + O_\varepsilon(x^\varepsilon)$ *for every* $\varepsilon>0$.

Keywords. Generalised Dirichlet series; distribution of zeros; neighbourhood of the critical line.

1. Introduction

This paper ought to have been paper XII of this series. But elsewhere [5] the second author has referred to this paper as paper XIV, because there are two new additions to this series namely, On the zeros of $\zeta'(s)-a$, (on the zeros of a class of generalised Dirichlet series-XII) and On the zeros of $\zeta(s)-a$, (on the zeros of a class of generalized Dirichlet series-XIII) both of which will appear in *Acta Arithmetica* with the short titles only. The addition elsewhere of the title in the brackets have been made only for some technical convenience. In the present paper we continue the investigations of the papers III [1], IV [2], V [4], and VI [3]. Just as VI [3], was in the nature of an addendum to the earlier papers, this note is a modest progress beyond the paper