Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 77–92.

On Fourier coefficients of Maass cusp forms in 3-dimensional hyperbolic space

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Dedicated to the memory of Professor K G Ramanathan

Abstract. In this article we establish the analogue of a theorem of Kuznetsov (theorem 6 of [3]) in the case of 3-dimensional hyperbolic space. We also consider a generalization of this result for higher dimensional hyperbolic spaces and discuss the relevant ingredients of a proof.

Keywords. Fourier coefficients, Maass cusp forms; 3-dimensional hyperbolic space; Kuznetsov theorem.

Let $\mathbb{H}_3 = \{w = z + jy \mid z = x_1 + ix_2 \in \mathbb{C}, x_1, x_2, y \in \mathbb{R}, y > 0, ij = -ji, j^2 = -1 = i^2\}$ be the 3-dimensional hyperbolic space. The group $PSL(2;\mathbb{C})$ acts on $\mathbb{H}_3$ via the mappings $w \to \gamma\langle w\rangle := (aw+b)(cw+d)^{-1}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2;\mathbb{C})$. If R denotes the ring of integers in an imaginary quadratic field K over $\mathbb{Q}$ of discriminant $d = d_K < 0$ in $\mathbb{Z}$, then $\Gamma := PSL(2;R)$ is a discontinuous group of homeomorphisms $w \mapsto \gamma\langle w\rangle$ of $\mathbb{H}_3$ onto $\mathbb{H}_3$, with a fundamental domain $\mathscr{F}$. On $\mathbb{H}_3$, we have the $PSL(2;\mathbb{C})$-invariant volume element $dv = y^{-3}dx_1 dx_2 dy$. Let $\Gamma_\infty := \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma\right\}$ and $\Gamma'_\infty := \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty\right\}$. For $w \in \mathbb{H}_3$ and $\gamma \in \Gamma$, we write $\gamma\langle w\rangle = z(\gamma\langle w\rangle) + jy(\gamma\langle w\rangle)$.

For n in $R^* := \{2a/\sqrt{d} \mid a \in R\}$, $w \in \mathbb{H}_3$ and s in $\mathbb{C}$ with $\mathrm{Re}(s) > 2$, the Poincaré series $P(w,s;n)$ is defined by

$$P(w,s;n) = \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma} y(\gamma\langle w\rangle)^s \exp[-2\pi|n|y(\gamma\langle w\rangle) + 2\pi i\,\mathrm{Re}(z(\gamma\langle w\rangle)\bar{n}.)]$$

For $n = 0$, $P(w,s;0)$ is precisely the Eisenstein series denoted by $E(w,s)$. While $P(w,s;n)$ is real analytic on $\mathbb{H}_3$ and invariant under Γ, it is in $L^2(\mathscr{F})$ only for $n \neq 0$; on the other hand, it is an eigenfunction of the Laplacian Δ on $\mathbb{H}_3$ only for $n = 0$, with eigenvalue $s(s-2)$.

The Eisenstein series $E(w,s)$ can be continued meromorphically for all s in $\mathbb{C}$, the only singularity being a simple pole at $s = 2$. The Poincaré series $P(w,s;n)$, for $n \neq 0$, have a meromorphic continuation for all $\mathrm{Re}(s) > 1$ and are holomorphic there except possibly at a finite number of points s_l; the finitely many $\mu_l := s_l(2 - s_l)$ in $(0,1)$ are called the 'exceptional' eigenvalues (in the spectrum of Δ). The Eisenstein series $E(w, 1+it)$ for t in $\mathbb{R}$ together with similar series for the cusps of $\mathscr{F}$ different from ∞ 'span' the continuous spectrum of Δ; further, within $[1,\infty)$, there is also the discrete