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Modular forms and differential operators

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Dedicated to the memory of Professor K G Ramanathan

Abstract. In 1956, Rankin described which polynomials in the derivatives of modular forms are again modular forms, and in 1977, H Cohen defined for each $n \geqslant 0$ a bilinear operation which assigns to two modular forms f and g of weight k and l a modular form $[f,g]_n$ of weight $k+l+2n$. In the present paper we study these "Rankin-Cohen brackets" from two points of view. On the one hand we give various explanations of their modularity and various algebraic relations among them by relating the modular form theory to the theories of theta series, of Jacobi forms, and of pseudodifferential operators. In a different direction, we study the abstract algebraic structure ("RC algebra") consisting of a graded vector space together with a collection of bilinear operations $[,]_n$ of degree $+2n$ satisfying all of the axioms of the Rankin–Cohen brackets. Under certain hypotheses, these turn out to be equivalent to commutative graded algebras together with a derivation ∂ of degree 2 and an element Φ of degree 4, up to the equivalence relation $(\partial, \Phi) \sim (\partial - \phi E, \ \Phi - \phi^2 + \partial(\phi))$ where ϕ is an element of degree 2 and E is the Euler operator ($=$ multiplication by the degree).

Keywords. Modular forms; Jacobi forms; pseudodifferential operators; vertex operator algebras.

The derivative of a modular form is not a modular form. Nevertheless, there are many interesting connections between differential operators and the theory of modular forms. For instance, every modular form (by which we shall always mean a holomorphic modular form in one variable of integral weight) satisfies a nonlinear third order differential equation with constant coefficients; in another direction, if such a form $f(\tau)$ is expressed as a power series $\varphi(t(\tau))$ in a local parameter $t(\tau)$ which is a meromorphic modular function of τ, then the power series $\varphi(t)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients, where k is the weight of f. This latter fact, which leads to many connections between the theory of modular forms and the theory of hypergeometric and other special differential equations, played an important role in the development of both theories in the 19th century and up to the work of Fricke and Klein, but surprisingly little role in more modern investigations.

In 1956, R. A. Rankin [Ra] gave a general description of the differential operators which send modular forms to modular forms. A very interesting special case of this general setup are certain bilinear operators on the graded ring $M_*(\Gamma)$ of modular forms on a fixed group $\Gamma \subset PSL(2, \mathbb{R})$ which were introduced by H. Cohen [Co] and which have had many applications since then. These operators, which we call the Rankin-Cohen brackets, will be the main object of study in the present paper. On the one hand, we will be interested in understanding from various points of view