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Multiplicative arithmetic of finite quadratic forms over Dedekind rings

ANATOLI ANDRIANOV*

Sonderforschungsbereich 170, "Geometrie und Analysis", Mathematisches Institut der Universität, Bunsenstr. 3–5, D-3400 Göttingen, Germany

*Permanent address: St. Petersburg Branch of the Steklov Mathematical Institute, Fontanka 27, 191011 St. Petersburg, Russia

Dedicated to the memory of Professor K G Ramanathan

Abstract. Let $q(X)$ be a quadratic form in an even number m of variables with coefficients in a Dedekind ring K. Let us assume that the sets

$$R(q,a)=\{N\in K^m;\, q(N)=a\}$$

of representations of elements a of K by the form q are finite. Then certain multiplicative relations are obtained by elementary means between the sets $R(q,a)$ and $R(q,ab)$, where b is a product of prime elements ρ of K with finite coefficients $K/\rho K$. The relations imply similar multiplicative relations between the numbers of elements of the sets $R(q,a)$, which formerly could be obtained only in some special cases like the case when $K=\mathbb{Z}$ is the ring of rational integers and only by means of the theory of Hecke operators on the spaces of theta-series. As an application, an almost elementary proof of the Siegel theorem on the mean number of representations of integers by integral positive quadratic forms of determinant 1 is given.

Keywords. Quadratic forms; multiplicative properties; rings of automorphs; Siegel theorem.

1. Introduction

We consider quadratic forms

$$q(X)=\sum_{1\leqslant i\leqslant j\leqslant m} q_{ij}x_ix_j,\quad \text{where}\quad X=\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}, \tag{1.1}$$

with coefficients q_{ij} in a Dedekind ring K and we shall be interested in relations between the sets

$$R(q,a)=R_K(q,a)=\{N\in K^m;\, q(N)=a\} \tag{1.2}$$

of the representations (over K) of various elements a of K by the form q. It turns out that under certain conditions on the form q and a principal prime ideal ρK of K each representation $N\in R(q,\rho a)$ can be factorized in the form

$$N=DN',$$