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The density of rational points on non-singular hypersurfaces

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Let $F(\mathbf{x}) = F[x_1,\ldots,x_n]\in\mathbb{Z}[x_1,\ldots,x_n]$ be a non-singular form of degree $d\geqslant 2$, and let

$$N(F,X)=\#\{\mathbf{x}\in\mathbb{Z}^n; F(\mathbf{x})=0, |\mathbf{x}|\leqslant X\},$$

where

$$|\mathbf{x}| = \max_{1\leqslant r\leqslant n} |x_r|.$$

It was shown by Fujiwara [4] [Upper bounds for the number of lattice points on hypersurfaces, *Number theory and combinatorics, Japan, 1984*, (World Scientific Publishing Co., Singapore, 1985)] that $N(F,X)\ll X^{n-2+2/n}$ for any fixed form F. It is shown here that the exponent may be reduced to $n-2+2/(n+1)$, for $n\geqslant 4$, and to $n-3+15/(n+5)$ for $n\geqslant 8$ and $d\geqslant 3$. It is conjectured that the exponent $n-2+\varepsilon$ is admissable as soon as $n\geqslant 3$. Thus the conjecture is established for $n\geqslant 10$. The proof uses Deligne's bounds for exponential sums and for the number of points on hypersurfaces over finite fields. However a composite modulus is used so that one can apply the 'q-analogue' of van der Corput's AB process.

Keywords. Rational points; hypersurface; counting function; multiple exponential sum; Deligne's bounds; singular locus.

1. Introduction

Let $F(\mathbf{x}) = F[x_1,\ldots,x_n]\in\mathbb{Z}[x_1,\ldots,x_n]$ be a non-zero form of degree d. We shall be concerned here with bounds for the number

$$N(F,X)=\#\{\mathbf{x}\in\mathbb{Z}^n; F(\mathbf{x})=0, |\mathbf{x}|\leqslant X\},$$

where

$$|\mathbf{x}| = \max_{1\leqslant r\leqslant n} |x_r|.$$

It is trivial that

$$N(F,X)\ll_F X^{n-1}.$$

Moreover it is clear that $N(F,X)\gg_F X^{n-1}$ whenever F has a rational linear factor. However in all other cases one has

$$N(F,X)\ll_F X^{n-3/2}\log X. \tag{1}$$