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On the equation $x(x+d_1)\ldots(x+(k-1)d_1) = y(y+d_2)\ldots(y+(mk-1)d_2)$

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Dedicated to the memory of Professor K G Ramanathan

Abstract. For given positive integers $m \geqslant 2, d_1$ and d_2, we consider the equation of the title in positive integers x, y and $k \geqslant 2$. We show that the equation implies that k is bounded. For a fixed k, we give conditions under which the equation implies that $\max(x, y)$ is bounded.

Keywords. Exponential diophantine equations; arithmetic-geometric mean.

1. Introduction

For positive integers $m \geqslant 2$, d_1 and d_2, we consider the equation

$$x(x+d_1)\ldots(x+(k-1)d_1) = y(y+d_2)\ldots(y+(mk-1)d_2) \tag{1}$$

in integers $x > 0$, $y > 0$ and $k \geqslant 2$. Equation (1) with $d_1 = d_2$ was considered in [4] and [5]. It was shown in [5, Corollary 2] that equation (1) with $d_1 = d_2 = d$ and $m \geqslant 2$ implies that $\max(x, y, k)$ is bounded by an effectively computable number depending only on m and d. In this paper, we extend this result as follows:

Theorem 1. *There exists an effectively computable number C depending only on d_1 and d_2 such that equation* (1) *with $m = 2$ implies that either*

$$\max(x, y, k) \leqslant C$$

or

$$k = 2,\ d_1 = 2d_2^2,\ x = y^2 + 3d_2 y.$$

On the other hand, we observe that equation (1) with $m = 2$ is satisfied whenever the latter possibility holds.

Theorem 2. *Let $m > 2$. Assume that equation* (1) *is satisfied. Then*
(a) *k is bounded by an effectively computable number C_1 depending only on m, d_1 and d_2.*
(b) *Let $k \leqslant C_1$. There exists an effectively computable number C_2 depending only on m, d_1 and d_2 such that either*

$$\max(x, y) \leqslant C_2 \tag{2}$$

or

d_1/d_2^m is a product of m distinct positive integers composed of primes not exceeding m.