J. Astrophys. Astr. (1999) **20**, 233–248

The Question of Cosmic Censorship*

Roger Penrose, *Department of Mathematics, University of Oxford, 24-29, St. Giles, Oxford 0X1 3BD, UK*
email: rouse@maths.ox.ac.uk

Abstract. Cosmic censorship is discussed in its various facets. It is concluded that rather little clear-cut progress has been made to date, and that the question is still very much open.

1. The role of cosmic censorship in gravitational collapse

Chandra's famous work on the maximum mass of white dwarf stars (Chandrasekhar 1931) pointed the way to our present-day picture (cf. Hawking & Penrose 1970) that for bodies of too large a mass, concentrated in too small a volume, unstoppable collapse will ensue, leading to a singularity in the very structure of space-time. The deduction of a strict occurrence of an actual singularity in physical space-time would, however, be based on an assumption that no quantum-mechanical principles intervene to change the nature of space-time from that which is classically described by Einstein's general relativity. Indeed, the term 'singularity', in this physical context, really refers to a region where the conventional classical picture of space-time breaks down, to be replaced, presumably, by whatever physics is to go under the name of 'quantum gravity'. It is the normal expectation that this occurs only when classical space- time curvatures diverge — quantum effects taking over when radii of space-time curvature of the order of the Planck length are attained.

In the standard picture of collapse to a black hole (of. Penrose 1978, for example), these singularities are not visible to observers at a large distance from the hole, being 'shielded' from view by an absolute event horizon. Thus, whatever unknown physics takes place at the singularity itself, its effects are not observable by such an observer. The assumption of cosmic censorship is that in a generic gravitational collapse the resulting space-time singularity will indeed be shielded from view in this way. Accordingly, it is taken that the alternative of a naked singularity – i.e. a visible singularity – would not occur (except, conceivably, for some very special initial collapse configurations which could not be expected to take place in an actual astrophysical circumstance).

It is not hard to conceive of physical situations in which one of the standard criteria for 'unstoppable gravitations collapse' is satisfied. All that is required is for sufficient mass to fall into a small enough region. For the central region of a large galaxy, for example, the required concentration could occur with the stars in the region still being separated from each other, so there is no reason to expect that there could be some overriding physical principle which conspires always to prevent such

* Reprinted with permission from the book 'Black Holes and Relativistic Stars', edited by R.M. Wald and published by the University of Chicago Press, Illinois, USA.

unstoppable collapse. However, we cannot simply deduce from this that a black hole will be the result. This deduction requires the crucial assumption that cosmic censorship, in some form, holds true.

Two familiar mathematical criteria for 'unstoppable collapse' are the existence of a trapped surface or of a point whose future light cone begins to reconverge in every direction along the cone. In either of these situations, in the presence of some other mild and physically reasonable assumptions, like the nonnegativity of energy (plus the sum of pressures), the nonexistence of closed timelike curves, and some condition of genericity (like the assumption that every causal geodesic contains at least one point at which the Riemann curvature is not lined up in a particular way with the geodesic), it follows (by results in Hawking & Penrose 1970) that a space-time singularity of some kind must occur. (Technically: the space-time manifold must be geodesically incomplete in some timelike direction.)

It appears to be a not uncommon impression among workers in the field that as soon as one of these conditions is satisfied – say the existence of a trapped surface- then a black hole will occur; and, conversely, that a naked singularity will be the result if not. However, it should be made clear that neither of these deductions is in fact valid. The deduction that a black hole comes about whenever a trapped surface is formed *requires* the assumption of cosmic censorship. Moreover, the deduction that some kind of space-time singularity comes about (in general situations), whether or not it is a naked one, requires some such assumption like that of the existence of a trapped surface (of. e.g. Hawking & Penrose 1970). Thus, the presence of a trapped surface does not imply the absence of naked singularities; still less does the absence of a trapped surface imply the presence of a naked singularity, These points have relevance to various investigations which attempts to address the issue of cosmic censorship by the study of specific examples. Frequently, the absence of a trapped surface appears to be regarded as a criterion for - or at least a strong indication of – cosmic censorship violation (cf. for example, Shaprio & Teukolsky 1991). It should be clear from the above remarks that the issue is by no means as simple as that.

While the question of cosmic censorship remains very much an open one at the present time – possibly the most important unsolved problem in classical general relativity – progress in certain areas has been made, and I shall attempt to address some of these in the following remarks. However, these should not be regarded as in any way a comprehensive survey of progress in these areas, but merely as a personal assessment of the present status of the subject.

2. Plausibility of criteria for unstoppable collapse

Before addressing the issue of cosmic censorship directly, it will be appropriate to make a few remarks concerning the question of whether the above criteria –namely, the existence of a trapped surface or of a reconverging light cone – actually do represent conditions that would be realized when too much mass is concentrated in too small a volume. Various researchers (most particularly Shoen & Yau 1993) have presented arguments to show that sufficiently large mass concentrations do indeed lead to the presence of trapped surfaces. I shall not attempt to summarize this area of work here, The arguments are mathematically quite difficult, but the physical implications of these arguments are still far from clear to me. On the other hand, the

reconverging light cone condition is easier to see as representing a plausible criterion. I shall have a few comments to make concerning this case. Basically, the argument for the physical realizability of the reconverging light cone condition is that given in Penrose (1969). Imagine a certain amount of massive material, say of total mass $\mathcal{M}$, and allow it to fall to within a roughly defined region whose diameter is of the general order of 4 GM. We consider a space-time point p somewhere in the middle of this region, and examine the future light cone $\mathcal{C}$ of p. Thus, $\mathcal{C}$ is swept out by the future-endless rays (null geodesies) with past endpoint p. The strict condition that $\mathcal{C}$ 'satisfies the reconverging light cone condition' would be that on every ray γ generating $\mathcal{C}$ there is a place where the divergence of the rays changes sign. If it is assumed that such a ray is geodesically complete in the future direction (and that the energy flux across the ray is nonnegative), then it follows that, to the future of p along the ray, there is a point *conjugate* to p (i.e. a point', distinct from p, with the property that there is a 'neighbouring ray to γ' which intersects γ to p and again at q; more precisely, there is a nontrivial Jacobi field along γ which vanishes both at p and at q). The idea is that as the material falls in across $\mathcal{C}$ it causes "focussing power" in the lensing effect of the Ricci tensor component along the ray (namely, $R_{ab}l^a l^b$ where l^a is a null tangent vector to γ) due to the energy density in the matter falling in across $\mathcal{C}$. There are simple integral expressions that can be written down (cf. Clarke 1993, in particular) which provide sufficient conditions for a conjugate point to arise, so it is merely an order-of-magnitude requirement that there is sufficient infall of material to ensure that the focussing condition will be satisfied. The situation could be made to be qualitatively similar to the original Oppenheimer-Snyder (1939) collapsing dust cloud (pressureless fluid), but where there is no symmetry assumed and no particular equation of stake employed (like that of Oppenheimer and Snyder's dust).

However, the strict form of the reconverging light cone condition is that *every* ray through p should encounter sufficient material for divergence reversal to occur. This condition might well be considered to be unreasonably strong. It might not be evidently satisfied if the collapsing material is concentrated in a number of separated bodies, say stars, rather that in a continuous medium. Many of the rays through p might then miss the actual collapsing material, so the focussing along such a ray could be much smaller than required (a point specifically raised with me by Robert Wald), being only a secondary effect due to the nonlocal overall focussing produced by Weyl curvature. In face, this does not make a serious difference, but to see that it does not is not entirely straight forward. The essential point is that for the purposes of the singularity theorem being appealed to here (Hawking & Penrose 1970), it is not necessary to assume that every future ray through p encounter divergence reversal. All that is required, roughly speaking, is that the elements of area of cross section of the intersection $\mathcal{C} \cap \partial I^+(p)$ of $\mathcal{C}$ with the boundary $\partial I^+(p)$ of the (chronological) future I^+ of p should eventually decrease in future directions, at every point of the cross section. At those places where a sufficient amount of the matter directly encounters 'crossing regions' of $\mathcal{C}$, where two parts of this null hypersurface encounter one another and both cross through into the interior of I^+ (so that they do not remain on the region of $\mathcal{C}$ that lies on $\partial I^+(p)$). To see that this must be the case when a sufficient concentration of material encounters $\mathcal{C}$, one may appeal to the qualitative similarity between the situations arising when the collapsing material consists of a continuous and fairly uniform medium, and when it consists of a number of discrete bodies (such as stars or, at a different level of description, the constituent

particles of the medium) which closely approximate that medium. This is sufficient for establishing that p constitutes a 'future-trapped set', in the sense that is required for Hawking & Penrose (1970), and the deduction of the presence of a singularity follows just as before. I shall not go into the details of this argument here, it being more appropriate to leave this for some later discussion. (It has a bearing on other matters that have been addressed in the literature, concerning the lensing effects of different kinds of mass distribution; c.f. Holz & Wald 1997).

3. Causal definition of naked singularities

Let us take it, then, that there will be certain astrophysical situations in which such 'unstoppable' gravitational collapse actually takes place—'unstoppable' in the sense that it leads to some kind of space-time singularity (in accordance with the singularity theorems). When this occurs, and assuming the form of cosmic censorship which assets that no such singularity is naked in the sense of being visible to some observer at infinity, then the boundary of the past of the 'set of observers at infinity' constitutes the *absolute event horizon.* External to this horizon are the space-time events which can send signals to infinity; cosmic censorship asserts that no singularity can lie within this external region. This presents us with the classical situation of a black hole.

Normally, the cosmic censorship issue is phrased in some such way, i.e. in terms of observers at large distances from the collapse — and, in precise mathematical discussions, in terms of observers at future null infinity $\mathcal{I}^+$. The absolute event horizon is then the boundary $\partial I^-[\mathcal{I}^+]$ of the past of $\mathcal{I}^+$. However, it may be felt that this is not necessary the appropriate notion of cosmic censorship. For one could imagine a situation in which an observer, near to a gravitationally collapsing body, witnesses a naked singularity arising, this singularity being visible to that particular observer. We might envisage that, in this universe model, the observer and the observed collapse are both within a large region of mass concentration which ultimately – perhaps on a cosmological timescale collapses to trap both the observer and observed region, thereby preventing signals from reaching 'infinity'. In such a picture, the singularity would not be 'naked' in terms of the definition which uses $\mathcal{I}^+$ since signals reaching the observer are themselves the singularity, so the singularity would indeed be naked in a more local sense. There being no theory governing what happens as the result of the appearance of such a singularity, this particular observer would not be able to account, in scientific terms, for whatever physical behaviour is seen. This is the kind of 'unpredictability' that one wishes to avoid in a physical situation. One of the reasons for desiring a cosmic censorship principle, after all, is the elimination of such physical uncertainties. A definition of 'naked singularity' which depends upon what is accessible to observers at infinity ($\mathcal{I}^+$) does not achieve this satisfactorily.

Thus, it seems not reasonable to posit a somewhat stronger form of cosmic censorship than one phrased in terms of observers at infinity. In accordance with this, an appropriate notion of *strong cosmic censorship* has been formulated (Penrose 1978) which has the added advantage that it turns out to be symmetrical in time. Moreover, for a given space-time $\mathcal{M}$ this notion turns out to be equivalent to *global hyperbolicity* for $\mathcal{M}$.

The condition can be phrased (see Geroch, Kronheimer, and Penrose 1972) in terms of the notion of *indecomposable pasts* – IPs – and *indecomposable futures* - IFs. An IP is a past-set (i.e. a subset of $\mathcal{M}$ which is the same as its own chronological past) which, in addition, is not the proper union of two other past-sets. An IF is defined correspondingly, with 'future' replacing 'past'. It can be shown (Geroch, Kronheimer & Penrose 1972) that the IPs are precisely the pasts of timelike (or causal) curves in $\mathcal{M}$ and the IFs are the futures of such curves. Such a timelike (or causal) curve is said to generate the IP and IF in question. An IP is called *proper* – a PIP – if it consists of the set of points lying to the chronological past of some point of $\mathcal{M}$ (which would be the future endpoint of such a generating curve). A PIF is, correspondingly, the (chronological) future of some point of $\mathcal{M}$. A terminal IP – a TIP – is an IP which is not a PIP; a terminal IF – a TIF – is an IF which is not a PIF. Sometimes, the TIPS and TIFs are called ideal points for the space-time $\mathcal{M}$.

Let us assume that $\mathcal{M}$ is *strongly causal* (i.e. that every point of $\mathcal{M}$ has an arbitrarily small neighbourhood such that no causal curve leaves and then reenters it; (cf. Hawking & Ellis 1973; Penrose 1972). In face, it is sufficient, in what follows; to assume that $\mathcal{M}$ is both *future distinguishing* – i.e. that no two points of $\mathcal{M}$ have the same chronological future) – and *past distinguishing* – i.e. that no two points of $\mathcal{M}$ have the same chronological past). Then the points of $\mathcal{M}$ are canonically in one-to-one correspondence with the PIPs of $\mathcal{M}$ and also in one-to-one correspondence with the PIFs of $\mathcal{M}$. he TIPs and TIFs of $\mathcal{M}$ namely, $\mathcal{M}$'s ideal points, provide the points of what is called the *causal boundary* $\mathcal{M}$ of $\mathcal{M}$ The *future* causal boundary $\partial^{+}\mathcal{M}$ of $\mathcal{M}$ is the set of TIPs of $\mathcal{M}$ and the *past* causal boundary $\partial^{-}\mathcal{M}$ is the set of TIFs of $\mathcal{M}$. As thus defined, $\partial\mathcal{M}$ is a disjoint union of $\partial^{+}\mathcal{M}$ and $\partial\mathcal{M}$ There are circumstances under which it may be felt that certain of the points of $\partial^{+}\mathcal{M}$ should be identified with certain of the points of $\partial\mathcal{M}$ (see Geroch, Kronheimer & Penrose 1972; such situations could be considered to represent violations of cosmic censorship). However, for the purpose of this article, I shall prefer to regard the two sets $\partial^{+}\mathcal{M}$ and $\partial\text{–}\mathcal{M}$ as being actually disjoint. The entire union $\partial\mathcal{M} \cup \partial\text{–}\mathcal{M} \cup \mathcal{M} = \partial\mathcal{M} \cup \mathcal{M}$ is the causal closure of $\overline{\mathcal{M}}$ of $\mathcal{M}$.

It is convenient to divide these ideal points into two classes, according to whether they are to represent *singular* points of $\mathcal{M}$ or *points at infinity* for $\mathcal{M}$. The simplest way to make such a distinction if to say that a TIP represents a point at (future) *infinity* – an ∞ -TIP–if it is generated by a timelike curve that is of infinite length into the future, and a TIF represents a point at (past) infinity – an ∞-bf TIF – if it is generated by a timelike curve that is of infinite length into the past. The remaining TIPs and TIFs – the singular TIPs and TIFs then represent the singular points of $\mathcal{M}$. Although the distinction between points as infinity and singular points appears to be the simplest, it is not always regarded as the most appropriate (cf. Clarke 1993 for example). According to the definition given here (taken from Penrose 1978) one can have a 'point at infinity' for which the space-time curvature diverges as that ideal point is approached. It might be reasonable to call such an ideal point a 'singular point at infinity', but it would be given by an ∞-TIP or ∞-TIF as defined here, nevertheless, and not by what I am referring to as a 'singular TIP' or 'singular TIF'.

Now, a naked singularity may be described as a singularity which lies to the future of some point of space-time, but which can also be 'seen' by some observer. The reason for the first part of this description is that one would not want some cosmic

censorship principle to exclude the big bang; that is to say, the big bang should not count as a 'naked singularity'. Thus, a naked singularity lies both to the future of some point $p \in \mathcal{M}$ and to the past of some other point $q \in \mathcal{M}$. In terms of TIPs, a *naked singular* TIP would be a singular TIP R which contains the point p and which lies to the past of a point q, i.e.

$$p \in R \text{ and } R \subset I^{-}(q) \text{ for some } p, q \in \mathcal{M}. \tag{1}$$

Here the standard notation $I^{-}(q)$ is used for the chronological past of a point 'correspondingly, $I^{-}[Q]$ stands for the chronological past of a *set* Q, and $I^{+}(q)$ and $I^{+}[Q]$ stand for the chronological futures of a point p and a set', respectively (notations already employed above). A point at infinity might also be 'naked', in the same sense, so we can define a naked ∞ — TIPR in just the same way. We can similarly define a naked singular TIF or ∞ – TIP's. TIF as a TIF S for which:

$$S \subset I^{+}(p) \text{ and } q \in S \text{ for some } p, q \in \mathcal{M}. \tag{2}$$

4. Strong cosmic censorship

Many of the reasons for wishing to exclude naked singular apply also to naked points at infinity. It is not better, from the point of view of uniqueness of evaluation, that a singularity. (Anti-de Sitter space in an example of a space-time possessing naked points at infinity; there are indeed reasons of this kind for regarding this model as 'unphysical'.) In any case, as was remarked upon above some ∞ – TIPs or ∞ – TIFs might arise from regions of infinite curvature and could be thought of as, in some sense, 'singular', in any case. Thus, it seems to be a reasonable formulation of the requirement that a space-time $\mathcal{M}$ be in accordance with cosmic censorship that there should be no naked TIPs in the above sense, whether they be singular TIPs or ∞ - TIPs. As was shown in Penrose (1979), this condition is *equivalent* to the condition that $\mathcal{M}$ be free of naked TIPs–by virtue of the fact that it is equivalent, also, to the condition that $\mathcal{M}$ be *globally hyperbolic*

The assertion that $\mathcal{M}$ is free of naked singularities in this sense can also be phrased in other equivalent ways. It is convenient to introduce causal relations between TIPs as follows. We say that the TIP P *causally precedes* the TIP Q if $P \subseteq Q$, and that the TIP P *chronologically precedes* the TIP Q if there exists a point $q \in Q$, such that $p \subset I^{-}(q)$. Then, by the above definiton, a TIP is naked if there is another TIP to its chronological future. Defining the causal relations between TIFs correspondingly if R and S are TIFs, R causally precedes S if $S\ S \subseteq R$ and R chronologically precedes S if there exists $r \in R$ such that $S \in I^{+}(\mathrm{r})$–we see that the TIF S is naked if there is another TIF to its chronological past. In this sense, naked singularities (or points at infinity) are *timelike* entities.

We can now formulate our principle of *strong cosmic censorship* as the assertion that naked singularities or points at infinity (in the above sense) do not occur in generic space-times, where it is assumed that Einstein's equations hold with some reasonable equations of state for the matter. This is still a little vague because of occurrence of the words 'generic' and 'reasonable' in the definition. In my opinion, it is probably not particularly helpful to try to be more precise at this stage. There is a fairly well defined intuitive meaning for the word 'generic'; no doubt, when the

appropriate theorem comes along, then an appropriately relevant definition of 'generic' will become clearer (as was the case with some singularity theorems; cf. Hawking & Penrose 1970). Without some such restriction, however, counterexamples to cosmic censorship can occur, such as in certain very special situations of spherically symmetrical collapse (cf. Christodoulou 1994; Choptuik 1993). As regards 'reasonable equations of state' the essential points is that we should exclude equations that could lead to singularities in generic situations even in special relativity (e.g. leading to infinite density), such as occurs with 'dust' when caustics arise in the flow lines. The equations of state should also be such as to ensure energy positivity (and perhaps stronger restrictions such as the dominant energy condition).

As yet, there is no mathematical theorem asserting the truth of any appropriate form of cosmic censorship in general relativity; yet, as we have seen above, cosmic censorship is an essential ingredient of the standard picture of gravitational collapse to a lack hole. As was indicated in section 5.2 above, one can present convincing arguments to show that situations can occur – and, indeed, *will* occur in appropriate circumstances of gravitational collapse when sufficient matter is being concentrated in too small a region in which singularities will arise according to general relativity. But without a cosmic censorship assumption, there is no guarantee that these singularities will not be naked. If the singular region is not naked, even merely in the weaker sense, that observers at *infinity* are not able to 'see' the singular region (example so that no ∞-TIP representing a point of $\mathcal{I}^+$ contains a naked singular TIP), then there will be some region of the space-time including the singularity region, that cannot be seen from $\mathcal{I}^+$. In other words, the causal past of $\mathcal{I}^+$ (written $J^-[\mathcal{I}^+]$ is not the whole of the space-time M so the chronological past $I^-[\mathcal{I}^+]$ cannot exhaust $\mathcal{M}$ either. As stated in section 5.3 above, the boundary ∂I defines the *horizon* of the resulting black hole.

There is another crucial role that is played by cosmic censorship in the theoretical discussions of gravitational collapse. Without such an assumption, one cannot deduce the well known *area increase* theorem (cf. Floyd & Penrose 1971; Hawking 1972). This theorem asserts that the areas of cross section of a black-hole horizon $\partial I^-[I^+]$ are nondecreasing into the future. There are various different versions of this theorem, depending upon which version of cosmic censorship is adopted (see Penrose 1978 for a discussion of several of these). The area-increase theorem is important, particularly because of its relation to black-hole entropy and thermodynamics (Bekenstein 1973; Hawking 1975).

5. Thunderbolts

One remark should be made here concerning censorship proposals of this nature. They do not, as they stand, eliminate the possibility of what Hawking (1993) refers to as *thunderbolts,* first considered in Penrose (1978). This is the hypothetical situation according to which a gravitational collapse results in a 'wave of singularity' coming out from the collapse region, which destroys the universe as it goes! On this picture, the entire space-time could remain globally hyperbolic since everything beyond the domain of dependence of some initial hypersurface is cut off ('destroyed') by the singular wave. An observer, whether at infinity or in some finite location in the space-time, is destroyed just at the moment that the singularity would have become visible,

so that observer cannot actually 'see' the singularity. One condition which excludes this particular possibility (Penrose 1978, conditions CC4) is

no ∞–TIP contains a singular TIP.

For an asymptotically flat space-time $\mathcal{M}$ we may expect that the future-null conformal boundary $\mathcal{I}^+$ of $\mathcal{M}$ can be identified with its set of ∞ –TIPs. In any situation where the above condition is violated, we have an ∞ – TIP which directly 'sees' the singularity (in the sense of being causally to its future). In the situation where a thunderbolt is present, the 'observer at infinity' represented by that ∞ – TIP would be destroyed by the wave of infinite curvative at that very moment, but we still have an ideal point there, represented by this ∞ – TIP. However, the conformal boundary $\mathcal{I}^+$ would cease to be smooth at that point.

In section 5.4, one formulation of the condition of strong cosmic censorship was given as an assertion that 'timelike' singularities (or points at infinity) are to be excluded. The above condition for ruling out thunderbolts can be phrased as the condition that a point at infinity cannot lie *causally* to the future of a singular point (defined in terms of TIPs). One could imagine formulating an 'extrastrong' version of cosmic censorship in which *all* causally separated (distinct) TIPs are excluded (in the sense that no TIP shall properly include another TIP; cf. section 5.4) and not merely the timelike separated ones that are excluded by ordinary strong cosmic censorship. However, this condition would be unreasonable because it would rule out all asymptotically flat space-times! Being a null hypersurface, the future of the conformal boundary $\mathcal{I}^+$ of an asymptotically flat $\mathcal{M}$ contains null generators, and any pair of TIPs representing two distinct points of the same generator would be causally separated in the above sense.

However, it might well be reasonable to expect that a slightly weaker extrastrong version of cosmic censorship might be appropriate, in which it is asserted that there is no pair of distinct TIPs P, Q such that $P \subset Q$, and where P is a singular TIP (and where the corresponding statement in terms of TIFs could also be appended, if desired). This would incorporate both strong cosmic censorship and the exclusion of thunderbolts, in the above sense. It remains to be seen whether such a formulation might still be too strong.

6. Some arguments against cosmic censorship

Most of the arguments presented to date which are aimed at disproving cosmic censorship have been concerned with the examination of specific examples. However, there is an inherent difficulty in using arguments of this kind, because any specific example that can be studied in detail is liable to be 'special' in some way or other, and unlikely to be considered to be 'generic' in some appropriate sense. At least, this applies to specific examples that can be studied analytically in detail. It may be that with the further development of numerical techniques and computer power, examples might eventually be considered which could indeed be argued to be appropriately 'generic'. On the other hand, there is the compensating difficulty that with numerical solutions there may be some doubt, in any particular case, whether a seeming singularity is actually a genuine singularity, or whether the singularity is indeed naked. As things stand, specific examples can only give *indications* as to whether cosmic censorship is likely to be true, not definitive answers.

The first example of a gravitational collapse leading to a naked singularity was that given by Yodzis, Seifert, & Muller zum Hagen (1973). They pointed out that even in exactly spherically symmetrical collapse, infinite-curvative naked singularities could arise with dust, owing to the presence of caustics in the family of dust world-lines (at which the density diverges), provided that these caustics occur before an absolute event horizon is reached. As suggested in section 5.4, such circumstances should not be considered as providing violations of cosmic censorship because the infinite densities that arise have nothing to do with general relativity (or, indeed with gravity at all) because they occur just as readily with the equations for dust in special relativity.

Such regions of infinite density are sometimes referred to as 'shell-crossing' singularities, since they are regions where the different shells of collapsing material begin to cross one another. However, this terminology is not altogether appropriate because the difficulties with infinite density do not occur in the regions where different dust flows actually cross each other, but at the boundary of such a region, where there is a caustic in the flow lines and one flow becomes three[1] Nevertheless, where there are indeed several superimposed flows, the energy momentum tensor of 'dust' cannot be used, but instead one has a sum of a number of different terms of this kind, i.e.

$$T_{ab} = \rho u_a u_b + \cdots + \tau w_a w_b, \tag{3}$$

where $\rho,\ldots,\tau$ are the respective densities of the different components of the dust (pressureless fluid) and where each of $u^a,\ldots,w^a$ is a unit future-timelike vector giving the direction its flow. For each component of the flow, the flow world-lines are geodesics, and each of ρu_a ,..., τwa is divergence-free. Of course, regions of infinite density can still arise whenever one of the systems of flow lines encounters caustics.

We can generalize the above finite sum of terms to a situation in which there is a *continuum* of terms. This gives us an instance of the kind of system that is treated by the *Vlasov* equation. More generally, the Vlasov equation covers the cases when there is a continuous superposition of fluids which possess pressure – rather than being just 'dust' as in the cases considered above (the simpler case of a 'collisionless' fluid).

In the collapse situation studied by Shapiro & Teukolsky (1991), referred to in section 5.1, the Vlasov equation is used but (as was pointed out to me by Alan Rendall) the individual fluid components do not possess pressure (the 'collisionless' case), and it is not clear that the situation is free of the problems that occur with the Yodzis, Seifert & Muller zum Hagen (1973) example. Building upon earlier ideas of Thorne, who suggested that prolate spheriodal collapse might lead to naked singularities (because of a resemblance to *cylindrical* trapped surface-free collapse; cf. also Thorne 1972; Chrusciel 1990), Shapiro and Teukolsky consider the collapse of a prolate azisymmetrical body composed of collisionless material, and they argue that naked singularities can arise. They indicate the presence of regions at which the density diverges and argue from the absence of trapped surfaces that these singularities could well be naked. Moreover, they point out that their singular regions do not arise merely from infinite density, because they extend outside the matter region.

[1] It does not become merely *two* overlapping flows, for topological reasons. One of the three flows 'counts' as negative and the other two as positive, preserving the net count of overlapping flows, as one passes from one side of the caustic to the other.

However, as was pointed out in section 5.1 above, more needs to be established if we are to ascertain whether these singularities are indeed naked. In particular, we would need to examine the regions of the space-time lying to the future of the singularity, but this is not possible within the framework of the computer calculation that they carry out, since the calculation terminates as soon as the singularity is reached.

Moreover, as was pointed out by Iyer & Wald (1991), collapse situations that appear to resemble that of Shapiro and Teukolsky can be constructed where no trapped surfaces appear before *nonnaked* singularities arise. In both the Iyer-Wald example and the Shapiro-Teukolsky example, there is a reasonable-looking family of constant-time spacelike hypersurfaces according to which the time evolution is described. In neither example are there trapped surfaces before a singularity appears. However, the singularity is clearly *not* naked in the Iyer-Wald example, because their example is simply the ordinary extended Schwarzschild solution described according to a nonstandard time coordinate. This sheds considerable doubt on the Shapiro-Teukolsky suggestion that their singularity is actually naked.

A closely related situation was studied by Tod (1992). In this example, there is a collapsing shell of 'null matter' (a delta-function shell of massless dust) which fall into a region of Minkowski space that it surrounds. The mass density can vary arbitrarily with spatial direction, and the (convex, smooth) shape of the shell, at one particular time, can also be chosen arbitrarily. By choosing this shape to be a suitable prolate ellipsoid it is not hard to ensure that caustics in the collapsing shell – and hence singularities – arise before there are any trapped surfaces. The description is given in terms of standard t = const. hypersurfaces in the interior Minkowski space. Nevertheless, the situation is completely consistent with the conventional picture of gravitational collapse to a black hole. Trapped surfaces do in fact occur in the space-time, but not until after the t-value at which singularities arise. This is again similar to the Shapiro-Teukolsky situation, and there is no reason to expect a violation of cosmic censorship.

Other examples have been described (Choptuik 1993: Christodoulou 1994) in which the collapsing matter consists of a massless scalar field. In some of these, there are naked singularities. However, all these examples are extremely special, owing to the fact that spherical symmetry is assumed. Accordingly, it is hard to see that such examples can shed a great deal of light on the general issue of cosmic censorship. The condition of genericity is far from being satisfied. Moreover, according to a recent result of Christodoulou (1997), 'almost all' examples, even within *this* limited class, are free of naked singularities.

It would thus appear that there is, so far, no convincing evidence against cosmic censorship's being a principle with which classical general relativity accords. Quantum general relativity, on the other hand, does raise some serious problems in this regard. It is hard to avoid the conclusion that the endpoint of the Hawking evaporation of a black hole would be a naked singularity – or at least something that one a classical scale would closely resemble a naked singularity. Nevertheless these considerations are not directly relevant to what is normally referred to as 'cosmic censorship', which is intended to be a principle applying to *classical* general relativity only. When quantum effects are allowed, negative energy densities are possible – needed for the consistency of the Hawking effect, in which the area-increase property for a black-hole horizon is violated. In any case, unless there are mini-black holes in the universe (and the observational evidence seems to be against his), there would seem to be no

direct astrophysical or cosmological role for the 'objects' which represent the final stages of Hawking evaporation, owing to the absurdly long timescales needed for this process when it originates with an astrophysical black hole. (For theoretical considerations, on the other hand, these 'objects' could well be important – but that is another story!)

7. Some arguments in favour of cosmic censorship

There being no convincing evidence against cosmic censorship, we must ask whether, on the other hand, there is any convincing evidence in favour of it. Indeed, there are no results that I am aware of which give direct and convincing support to the view that there is a mathematical theorem asserting some form of cosmic censorship in classical general relativity. But are there any plausible general lines of argument aimed in this direction? I am not sure. In Penrose (1979, pp. 625–626), I put forward some rather vague suggestions of this nature, but, to my knowledge, these have not been followed up in a serious way. The idea was to try to show, roughly speaking, that Cauchy horizons are unstable, in some appropriate sense – at least for an initial Cauchy hypersurface Σ which is either compact or appropriately asymptotically flat. The idea would be that in the 'generic' case, the Cauchy horizon $H^+(\Sigma)$ would be replaced by a singularity, so that the maximal space-time consistent with evolution from Σ would in fact be the domain of dependence of Σ. This would have to be globally hyperbolic, i.e satisfy strong cosmic censorship. There is some evidence for such an instability (for asymptotically flat Σ) in work which shows that the 'inner horizon' (Cauchy horizon) of the Reissner-Nordström space-time (and of the Kerr space-time) is unstable (owing to the occurrence of infinitely blueshifted radiation); cf. Simpson & Penrose (1973), McNamara (1978a, 1978b), and Chandrasekhar & Hartle (1982). For further references concerning the issue of the (in)stability of black-hole Cauchy horizons in general relativity, see Ori (1997) and the article by Israel (chap. 7) in this volume.

On the other hand, there is some evidence that when there is a positive cosmological constant in Einstein's equations, stable Cauchy horizons may be possible. This situation comes about when the surface gravity of the cosmological horizon is greater than that of the Cauchy horizon which can occur with Reissner-Nordstrom-de Sitter and Kerr-de Sitter space-times (see Chambers & Moss 1994; cf. also Mellor & Moss 1990,1992, Brady & Poisson 1992). This situation is closely related to that considered below, in which inequalities arise from 'dropping particles into black holes'. As suggested below, it may well be that cosmic censorship requires a zero (or at least a nonpositive) cosmological constant.

Even if such a general result could be proved, it is not clear to me that this would really establish what is required for a suitable cosmic censorship theorem. It would not seem to rule out the thunderbolts discussed in section 5.5. This would really be necessary in order that the standard picture of gravitational collapse to a black hole can be obtained. What is the theoretical evidence that this picture is indeed likely to be always the correct one, according to classical general relativity? There seems to be little direct mathematical evidence. There are, however, certain rigorous mathematical results that give *indirect* support for cosmic censorship in this form. Ironically, these results have come about from a specific attempt to *disprove* cosmic censorship!

In Penrose (1973) I put forward a family of examples of gravitational collapse in which there is collapsing spherical shell of null dust, the density being an arbitrary function of direction out from the centre, the region inside the shell being Minkowski space. Shortly afterwards, Gibbons (1972) pointed out that there is no need for the shell to be spherical, and he considered this more general case of a smooth convex collapsing shell of null dust which surrounds a region of Minkowski space. (This is the generalization employed by Tod, 1992, referred to in section 5.6 above). As the shell collapses inwards, the matter density (the coefficient of a delta function) increases inversely as the area of cross section of the shell until it gets to a point where it can reverse the divergence of an intersecting outgoing light flash that originates in a region within the Minkowski space. If it does this all the way around, then the intersection S of that light flash with the infalling matter shell will be a trapped surface in the region just beyond the shell. All the geometry that needs to be considered for this takes place in Minkowski space. It depends only on the shapes of the collapsing shell and outgoing light flash, and on the matter density distribution on the shell.

Suppose that, in some particular shell geometry and matter distribution, it is possible to find an outgoing light flash for which S does provide us with a trapped surface. Then, according to the standard picture of collaspe to a black-hole - of which cosmic censorship is the most contentious part - the space-time will settle down to become a Kerr space-time in future asymptotic limit. If we assume this to be the case, we find that a certain geometrical inequality must hold true. Suppose the area of S is A_0, the area of the intersection of the absolute event horizon $\partial I^-[I^+]$ with the infalling matter shell is A_1, the future limit of the area of cross section of the (Kerr) horizon is A_2, and the area of the horizon of a Schwarzschild black hole of the same mass m is $A_3 = 16\pi m^2$ (units such that $G = c = 1$). We then have

$$A_0 \leq A_1 \leq A_2 \leq A_3 \leq 16\pi m_0^2, \tag{4}$$

where m_0 is the rest mass of the total energy-momentum of the incoming null dust shell[2].

The first inequality follows from the fact that the shell is infalling; the second follows from the area-increase theorem (which, as we recall, requires cosmic censorship); the third follows because the area of the Kerr horizon is smaller than that of Schwarzschild for the same mass; the fourth is a consequence of $m \leq m_0$, a relation which expresses the fact that, although there might be a loss of mass due to gravitational radiation, there will not be a gain (because of the Bondi-Schis mass-loss theorem and the asymptotic Minkowskian traingle inequality), the radiation being assumed to be entirely outgoing. Note that, in addition to cosmic censorship, there are (reasonable, but unproved) assumptions that the black hole actually *settles down* to become a Kerr space-time in the asymptotic limit (the known theorems simply assumeing stationarity) and that the usual asymptotic assumptions for asymptotically flat space-times hold good (both at spacelike and null infinity).

The two quantities A_0 and m_0 depend only on the initial Minkowski space setup. If any such example could be found for which $A_0 > 16\pi m_0^2$, then this would provide a counterexample to the standard picture of gravitational collapse-essentially contra-

[2] That this is the rest mass of the incoming shell is a point that was glossed over in Penrose (1973).

dicting cosmic censorship. However, no example of this kind has ever been constructed. Moreover, various versions of the inequality $A_0 \leq 16\pi m_0^2$ have been proved by different authors (some of which refer to a somewhat different situation in which the geometry within a spacelike hypersurface is used); see Gibbons (1972, 1984, 1997) Jang and Wald (1977), Geroch (1973), and Huisken & Ilmanen (1997). Although none of these results directly established any form of cosmic censorship, they may be regarded as offering it some considerable support. Cosmic censorship could be said to supply a behind-the-scenes 'reason' why these inequalities are true!

There are also other types of inequalities which have been regarded as 'tests' of cosmic censorship. One may ask the question whether it is possible to 'spin up' a Kerr (or Kerr-Newman) black hole to a degree where its angular momentum exceeds the value for which a horizon is possible, by allowing particles to drop into it. The mass, angular momentum, and charge of the particles come into the calculation, and various inequalities relate these to the black hole's geometrical parameters, in order that the horizon be preserved. It appears that these inequalities are always satisfied (cf. for example, Wald 1974; Semiz 1990) — except, curiously, if there is a positive cosmological constant (a situation pointed out to me by Gary Horowitz; cf. Brill *et al* 1994) I am not sure of the significance of this final proviso. Of course, it might be the case that cosmic censorship requires a zero cosmological constant. We recall from section 5.4 that a negative cosmological constant (in anti-de Sitter space) leads to naked points at infinity. It is not at all inconceivable that a positive cosmological constant might correspondingly lead to naked singular points. This has relation to the issue of the instability of Cauchy horizons, as noted above.

The question of whether a black-hole horizon can be 'destroyed' by perturbing it with Mailing matter is really part of the general question of the stability of a black-hole horizon. An unstable horizon could be expected to lead to a naked singularity. As far as I am aware, the arguments that have been given for horizons stability are fairly firm, but not yet fully conclusive. It would be interesting to know whether the presence of a cosmological constant makes a significant difference. My own feelings are left somewhat uncertain by all these considerations.

8. Do we need new techniques?

As will be seen from the preceding remarks, we are still a long way from any definite conclusions concerning cosmic censorship. It is possible that radically new mathematical techniques will be required for any real progress to be made. My particular preferences would be for techniques related to developments in *twistor theory*. At the most immediate level, twistor theory is concerned with the geometry of the space $\mathbb{P}\mathcal{T}$ of rays (null geodesics) in a space-time $\mathcal{M}$ The points of $\mathcal{M}$ would be regarded as secondary structures, and those of $\mathbb{P}\mathcal{T}$ as being somewhat more fundamental. A point of $\mathcal{M}$ is interpreted, in $\mathbb{P}\mathcal{T}$,in terms of the family of rays through that point. This family has the structure of a sphere in $\mathbb{P}\mathcal{T}$ -in fact, a *Riemann* sphere, which is a 1-dimensional complex manifold. The central idea of twistor theory is to call upon the power of *complex analysis* (physically, because links with quantum mechanics). For this purpose, the 5-dimensional manifold $\mathbb{P}\mathcal{T}$ must be thought of in terms of a larger complex manifold which, in the case when $\mathcal{M}$ is Minkowski space, turns out to be complex projective 3-space.

There are many problems, as yet unsolved, associated with how twistor theory is to be applied to general (vacuum) space-time, and it will be a long time before it has anything serious to say about cosmic censorship. Nevertheless, it has found a large number of applications (see, in particular, Bailey & Baston 1990; Mason & Woodhouse 1996). So far, it has not been significantly used to treat global questions in general relativity. The closest it has come to this is in the work of Low (1990, 1994), where some progress is made towards the understanding the causal structure of a space-time in terms of linking properties of spheres (or of loops, in the case of a space-time of 2 + 1 dimensions) in this space of rays.

In relation to this, it may be noted that there is a connection between cosmic censorship and the topology of $\mathbb{P}\mathcal{T}$ if $\mathcal{M}$ satisfies strong cosmic censorship (i.e is globally hyperbolic), then the space $\mathbb{P}\mathcal{T}$ is *Hausdorff,* whereas it is not Hausdorff in many cases where cosmic censorship fails. For any real progress to be made towards applying twistor theory to questions such as cosmic censorship, however, some major advances in understanding how the Einstein (vacuum) equations relate to twistor theory are needed. There does seem to be a deep connection between twistor theory and the Einstein equations, however – as yet elusive. This link is mediated through the equations for helicity 3/2 massless field (Penrose 1992). It has been known for some time that the consistency conditions for such fields (in potential form) are the Einstein vacuum equations (Buchdahl 1958; Deser and Zumino 1976 and Julia 1982). The other end of the link is the fact that the space of charges for such fields in *Minkowski* space is *twistor* space. Bringing together all the facets of this connection has proved to be a difficult problem (see Penrose 1996).

Although twistor theory remains a long way from addressing any significant issues of cosmic censorhip, it does have relevance to various issues connected with general relativity and space-time geometry (see Huggett & Tod 1985; Penrose & Ridler 1986; Penrose 1996). Perhaps it already has something to say about cosmology. The picture of a big bang leading to a Friedmann-type universe with negative spatial curvative and hyperbolic spatial geometry fits in well with the complex analytic (Riemann sphere) underlying philosophy, while the flat and closed spatial geometries do not do nearly so well (see Penrose 1997). Although negative spatial curvature cannot really be said to be a 'prediction' of the theory, it is perhaps the nearest to one, in general relativity and cosmology, that the theory has yet come up with.

Acknowledgements

I am particularly grateful to Robert Wald for various important remarks and for his help with the reference. I am also grateful to the NSF for support under contract PHY93-96246.

References

Bailey, T. N., Baston., eds 1990, Twistors in Mathematics and Physics, London Mathematical Society Lecture Notes Series, No. 156 (Cambridge University Press, Cambridge).

Bekeristein, J. 1973, *Phys. Rev.*, **D7**, 2333.

Brady, P. R, Poisson, E. 1992, *Class. Quant. Grav.*, **9**, 121.

Brill, D. R., Horowitz, G. T., Kastor, D, Traschen, J. 1994, *Phys. Rev.*, **D49**, 840.

Buchdahl, H. A. 1958, *Nuovo Cim*., **10**, 96.
Chambers, C. M., Moss, I. G. 1994, *Class. Quant. Grav.*, **11**, 1035.
Chandrasekhar, S. 1931, *Astrophys. J*., **74**, 81.
Chandrasekhar, S., Hartle, J. B. 1982, *Proc. R. Soc. London.*, **A384**, 301.
Christodoulou, D. 1994, *Ann. Math*., **140**, 607.
Christodoulou, D. 1997. The instabilities of naked singularities in the gravitational collapse of a scalar field (to appear).
Choptuik, M. 1993, *Phys. Rev. Lett*, **70**, 8.
Chrusciel, P. 1990, *Ann. Phys*., **202**, 100.
Clarke, C. J. S. 1993, The Analysis of space-time singularities, Cambridge Lecture Notes in Physics (Cambridge University Press, Cambridge).
Deser, S., Zumino, B. 1976, *Phys. Lett*., **B62**, 335.
Floyd, R. M., Penrose, R. 1971, *Nature*, *Phys. Sci*, **229**, 117.
Geroch, R. 1973, *Ann. N.Y. Acad. Sci*, **224**, 108.
Geroch, R, Kronheimer E. H., Penrose, R. 1972, *Proc. R. Soc. London.*, **A347**, 545.
Gibbons, G. W. 1972, *Commun. Math. Phys*., **27**, 87.
Gibbons, G. W. 1984, in Global Riemannian Geometry (eds) T. Willmore and N. J. Hitchin (Ellis Horwood, Chichester).
Gibbons, G. W. 1997, Collapsing shells and the isoperimetric inequality for black holes, hel-th/9701049.
Hawking, S. W. 1972, *Commun. Math. Phys*., **25**, 152.
Hawking, S. W. 1975, *Commun. Math. Phys*., **43**, 199.
Hawking, S. W. 1993, in The rennaissance of General relativity (in honour of D. W. Sciama), (eds) G. Ellis, A. Lanza and J. Miler (Cambridge University Press, Cambridge.)
Hawking, S. W., Ellis, G. F. R. 1973, The large scale structure of space-time (Cambridge University Press, Cambridge.)
Holz, D. E., Wald, R. M. 1997, A new method for determining cumulative gravitational lensing effects in inhomogenous universes, *Phys. Rev. D.* (Submitted).
Huggett, S. A., Tod, K. P. 1985, An introduction to Twistor Theory, London Mathematical Society Student Texts (L.M.S., London)
Huisken, G., Ilmanen, T. 1997, Proof of the Penrose inequality (to appear).
Iyer, V., Wald, R. M. 1991, *Phys. Rev*., **D44**, 3719.
Jang, P. S., Wald, R. M. 1977, *J. Math.*, *Phys*., **18**, 41.
Julia, B. 1982, *Comptes Rendus Acad. Sci.* Paris 295, Ser II, 113.
Low, R. 1990, *Class, Quant. Grav.*, **7**, 177.
Low, R.1994, *Class, Quant, Grav*., **11**, 453.
Mason, L. J., Woodhouse, N. M. J. 1996, Integrability, self-duality and twistor theory (Oxford University Press, Oxford).
McNamara, J. M. 1978a, *Proc. R. Soc. London*, **A358**, 499.
McNamara, J. M. 1978b, *Proc. R. Soc. London*, **A364**, 121
Mellor, M., Moss, I. G. 1990, *Phys. Rev.*, D41, 403.
Mellor, M., Moss, I. G. 1992, *Class. Quant. Grav.*, **9,** L43.
Oppenheimer, J. R., Snyder, H. 1939, *Phys. Rev.*, **56**, 455.
Ori, A. 1997, *Gen. Rel.Grav.*,**29**, 881.
Penrose, R. 1969, *Rivista del Nuovo Cim. Numero speciale* **1**, 252.
Penrose, R. 1972, Techniques of differential topology in relativity, CBMS Regional conf. ser. in Apl. Math., No. 7 (S.I.A.M., Philadelphia).
Penrose, R. 1973, *Ann. N. Y. Acad. Sci.*, **224**, 125.
Penrose, R. 1978, in Theoretical principles in astrophysics and relativity, eds. N. R. Liebowitz, W. H. Reid, P. O. Vandervoort (University of Chicago Press, Chicago)
Penrose, R. 1992, in General relativity: an Einstein centenary survey, (eds) S. W. Hawking and W. Isreal (Cambridge University Press, Cambridge)
Penrose, R. 1992, in Gravitation and Modern Cosmology (eds) A. Zichichi, N. de Sabbata and N. Sanchez (Plenum Press, New York).
Penrose, R.1996, in Quantum gravity: International school of cosmology and Gravitation, XIV Course, (eds) P. G. Bergmann, V. de Sabbata and H. J. Treder (World Scientific, Singapore).

Penrose, R., Rindler, W. 1996, Spinors and Space-time, vol. 2 Spinor and Twistor methods in space time geometry (Cambridge University Press, Cambridge)
Shapiro, S., Teukolsky, S. A. 1991, *Phys. Rev. Lett.*, **66***,* 994.
Schoen, R., Yau, S. T. 1993, *Commun. Math*, *Phys*., **90**, 575.
Semiz, I. 1990, *Class, Quant. Grav*., **7**,353.
Simpson, M., Penrose, R. 1973, *Int. J. Theor. Phys.*, **7**, 183.
Thorne, K. S. 1972, in Magic without Magic, ed. J. R. Klauder (Freeman, San Francisco).
Tod, K.P.1992, *Class.Quant.Grav.*, **9**, 1581.
Wald, R. M. 1974, *Ann. Phys.*, **82**, 548.
Yodzis, P., Seifert, H. J., Muller zum Hagen, H. 1973, *Commun. Math. Phys*., **34**, 135.