

LETTERS TO THE EDITOR

A NOTE ON THE ORTHOGONAL POLYNOMIALS

IN a previous paper, the author<sup>1</sup> proved two theorems concerning a Rodrigue's formula. Theorem (b) of that note admits of an extension which is presented here. Along with this, two theorems extending the recent work<sup>2</sup> of the author are also presented.

The notations are the same as in the previous papers:

*Theorem (a).*—The degree of  $X$  (an integer) which makes  $p_m(n, x)$ , an orthogonal set is  $(2m/n)$  where  $n$  may be fractional.

*Proof.*—Let  $X$  be of degree  $r$  (an integer) and let all the derivatives up to and including  $(m-1)$ th of  $(\omega X^n)$  be zero at  $(\alpha, \beta)$ . Consider

$$\begin{aligned} &\langle p_m(n, x), p_{m'}(n', x) \rangle \\ &= \int_{\alpha}^{\beta} \omega(x) p_m(n, x) p_{m'}(n', x) dx \\ &= \frac{(-1)^{m'}}{k_{m', n'}} \int_{\alpha}^{\beta} \omega X^{n'} \left( \frac{d}{dx} \right)^{m'} p_m(n, x) dx \\ &= 0 \text{ or a constant if } m' \leq (nr - m) \end{aligned}$$

since by cor. 1 or Theorem (a) of the note,<sup>1</sup>  $p_m(n, x)$  is of degree  $(nr - m)$ .

Similarly

$$\langle p_m(n, x), p_{m'}(n', x) \rangle = 0$$

or a constant if

$$m \leq (n'r - m')$$

These two conditions are together compatible if

$$nr = m + m'$$

$$n'r = m + m'$$

and hence

$$n = n'$$

and

$$r = \frac{m + m'}{n}$$

This as yet does not indicate the orthogonality of the set  $\{p_m(n, x)\}$ . To make it orthogonal, one desires  $p_m(n, x)$  to be of the precise degree  $m$ . Then,  $nr - m = m$  or  $nr = 2m$ .

Hence also  $m = m'$  in the above.

Therefore

$$\langle p_m(n, x), p_{m'}(n', x) \rangle = \Lambda_{m, n} \delta_{m, m'}$$

and  $r = 2m/n$  is an integer. This proves the theorem.

We observe that Theorem (b) of the note<sup>1</sup> referred to above is a special case of this theorem when  $m = n$  so that  $r = 2$ .

In view of this theorem, some transforms analogous to the ones suggested recently<sup>2</sup> for the case  $r = 2$  can be defined. Two theorems concerning them are stated without proof here, as they are quite similar to those in the work referred to above.

Defining either of the sequences of numbers

$$f_1(m, n) = T_1 \{f(x)\} = \int_{\alpha}^{\beta} \omega(x) p_m(n, x) f(x) dx$$

and

$$f_2(m, n) = T_2 \{f(x)\} = \int_{\alpha}^{\beta} p_m(n, x) f(x) dx$$

as the transforms of  $f(x)$ ,

*Theorem (b).*—Let  $f''(x)$  be bounded and integrable in any closed sub-interval of  $(\alpha, \beta)$ . Then if

$$\lim_{n \rightarrow \alpha, \beta} \omega(x) (X(x))^{m-n+1} f(x) = 0$$

$$\lim_{n \rightarrow \alpha, \beta} \omega(x) (X(x))^{m-n+1} f'(x) = 0,$$

then

$$T_1 \left\{ \frac{1}{\omega(x)} \left( \frac{d}{dx} \right) \left[ \omega(x) (X(x))^{m-n+1} f'(x) \right] \right\} = T_1 \{A(x) f(x)\}$$

where

$$A(x) = (X(x))^{m-n} \{ (m+1) (\frac{1}{2} m - n) X^{(n)} + nh^{(n)} - (m-n) (\log \omega)^{(n)} X \}$$

Or if

$$\lim_{n \rightarrow \alpha, \beta} (X(x))^{m-n+1} f(x) = 0$$

$$\lim_{n \rightarrow \alpha, \beta} (X(x))^{m-n+1} f'(x) = 0,$$

then

$$\begin{aligned} T_2 \left[ \left( \frac{d}{dx} \right) \left[ (X(x))^{m-n+1} f^{(n)}(x) \right. \right. \\ \left. \left. + (X(x))^{m-n} h(x) f(x) \right] \right] \\ = (X(x))^{m-n} h(x) f(x) \Big|_{\alpha}^{\beta} \\ = T_2 \{A(x) f(x)\}, \end{aligned}$$