LETTERS TO THE EDITOR

A GENERALISATION OF LAPLACE'S TRANSFORM

1. Transforms of the type

$$\phi(p) = p\int_0^\infty F(xp)\, f(x)\, dx$$

have long been known and the special case of this when $F(xp) \equiv e^{-xp}$ and hence

$$\phi(p) = p\int_0^\infty e^{-xp} f(x)\, dx \qquad (1)$$

has led to the subject of Operational Calculus, a powerful tool for tackling problems in Mathematical Physics and on which exists a good deal of literature. The relation (1) is known as Laplace's integral. It is possible to generalise[1] Laplacian Transform and the object of this note is to give an interesting generalisation and to introduce a new calculus based on this generalised Laplace's Transform.

2. If we take

$$F(xp) \equiv (2\,xp)^{-\frac{1}{2}} W_{k,m}(2\,xp)$$

where[2] $W_{k,m}(x)$ denotes a Whittaker Function, we obtain the transform

$$\phi_m^k(p) = p\int_0^\infty (2\,xp)^{-\frac{1}{2}} W_{k,m}(2\,xp)\, f(x)\, dx \qquad (2)$$

For $k = \frac{1}{2}$ and $m = \pm\frac{1}{4}$ when $(2xp)^{-\frac{1}{2}} W_{k,m}(2xp)$ degenerates into e^{-xp} we get Laplace's Transform.

We shall call $\phi_m^k(p)$ as the Whittaker or $W_{k,m}$-transform of $f(x)$ and $f(x)$ the original of $\phi_m^k(p)$ in this new transform.

In view of the involved nature of the function $(2xp)^{-\frac{1}{2}} W_{k,m}(2xp)$ as compared to the exponential function e^{-xp}, theorems based on this generalised Laplace's transform are not so easy to prove. We give, for the present, without proof, five theorems for this new Calculus.

Theorem 1A. If $\phi^k_m(p)$ is the Whittakar Transform of $f(x)$, then $\left(-p\frac{d}{dp}\right)^n \phi^k_m(p)$ is the Whittaker Transform of $\left(x\frac{d}{dx}\right)^n f(x)$.

Theorem 2A. If $\phi_m^k(p)$ and $\psi^{k'}_{m'}(p)$ are the $W_{k,m}$-and $W_{k',m'}$-transforms of $f(x)$ and $\psi(x)$ respectively, then

$$\int_0^\infty \phi^k_m(x)\,\psi(x)\frac{dx}{x} = \int_0^\infty \phi^{k'}_{m'}(x)\, f(x)\frac{dx}{x}.$$

This theorem may be considered as a Parseval Theorem in the Theory of Whittaker Transforms.

Theorem 1B. If $\phi^k_m(p)$ is the Whittaker transform of $f(x)$, then

$$\int_0^\infty \phi^k_m(x)\frac{dx}{x} = \frac{\Gamma(\frac{1}{2}-m)\,\Gamma(\frac{1}{2}+m)}{2\,\Gamma(\frac{1}{2}-k)}$$