ON FRACTIONAL REPLICATION OF THE GENERAL SYMMETRICAL FACTORIAL DESIGN

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FACTORIAL designs have now come to be extensively used for testing in a single experiment the effect of a number of interacting sets of factors. When, however, the number of sets of factors and/or the number of levels of each set of factors is large, there result a great number of treatment combinations. In such cases, even a single replication necessitates the use of large quantities of experimental material, the availability of which may be beyond the resources of the experimenter. It, therefore, becomes necessary to resort to the device of fractional replication which enables a factorial experiment to be carried out with only a fraction of the experimental units required for a complete replication. The basic principles of this theory have been developed in a recent paper by Finney,[1] who has, however, restricted himself to factorial arrangements p^n, where p is a prime number. It is the purpose of this note to show that, with the help of the geometrical theory of confounding given earlier by Bose and Kishen,[2] Finney's theory can be easily extended to the general symmetrical factorial arrangement s^m, where $s = p^n$, p being a prime positive integer and n any positive integer.

Bose and Kishen[2] have, by representing a treatment combination in an s^m factorial arrangement by a finite point of the associated m-dimensional projective geometry PG (m, s) constructed from the Galois field GF(s), established a (1, 1) correspondence between the s^m treatment combinations and the s^m points of the Euclidean geometry EG(m, s), which is a portion of PG(m, s). It has been demonstrated by Carmichael that the elements of the Abelian group of order s^m and type (1, 1, ..., 1) afford concrete representations of the EG(m, s). Thus, the correspondence of the s^m factorial design to the Abelian group of order s^m and type (1, 1, ..., 1) follows.

The treatment combinations in an s^m factorial arrangement may be represented by symbols $a_1^{\beta_1} a_2^{\beta_2} \ldots a_m^{\beta_m}$, where $\beta_1, \beta_2, \ldots, \beta_m$ take only the values $0, 1, 2, \ldots s-1$. If now $0, 1, 2, \ldots, s-1$ are taken to denote the s elements $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots \alpha_{s-1}$ respectively of GF (s), it would appear that these symbols form an Abelian group of order s^m and type (1,1 ...,1). It may be remarked here that there are different ways of identifying $\alpha_1, \alpha_2, \ldots \alpha_{s-1}$ with the $s-1$ non-zero elements of GF(s) when expressed in the standard form. In the case $n = 1$, *i.e.* when s is a prime number p, the identification we adopt is to take α_i equal to the residue class (i), modulo p. In the case when $n > 1$, *i.e.* when s is a power of a prime higher than the first, we take α_i equal to the residue class modulo $f(x)$ of the polynomial x^{i-1}, where $f(x)$ is a specified minimum function and the class with standard representative x is a primitive element of GF(s). The Abelian group of main effects and interactions, which is isomorphic to the group of treatment combinations, is then represented by the symbols $A_1^{\beta_1'} A_2^{\beta_2'} \ldots A_m^{\beta_m'}$, where $\beta_1', \beta_2', \ldots \beta_m'$ take only the values $0, 1, 2, \ldots, s-1$, these being the s elements of GF(s). Two elements $a_1^{\beta_1} a_2^{\beta_2} \ldots a_m^{\beta_m}$ and $a_1^{\beta_1'} a_2^{\beta_2'} \ldots a_m^{\beta_m'}$ of the treatment group will be defined to be orthogonal if $\Sigma \beta_i \beta_i' = 0$ in GF(s). Similarly, two elements $a_1^{\beta_1} a_2^{\beta_2} \ldots a_m^{\beta_m}$ and $A_1^{\beta_1'} A_2^{\beta_2'} \ldots A_m^{\beta_m'}$, the first of the treatment group and the second of the effect group, will be termed orthogonal if $\Sigma \beta_i \beta_i' = 0$ in GF(s). It would appear that if a treatment subgroup of order s^{m-k} is selected, the complete orthogonal effect subgroup is of order s^k.

The correspondence between effect subgroups and parallel pencils of $(m-1)$-flats representing main effects and interactions follows readily from Bose and Kishen's theory. Thus, the pencil $x_{i_1} = \alpha_j$ $(j = 0, 1, \ldots s-1)$ of s parallel finite $(m-1)$-flats representing the $s-1$ degrees of freedom for the main effect A_{i_1}, corresponds to the effect subgroup $I, A_{i_1}, A_{i_1}^2, \ldots A_{i_1}^{(s-1)}$ of order s, and the complete orthogonal treatment subgroup of order s^{m-1} is given by the s^{m-1} treatment combinations corresponding to the s^{m-1} finite points lying on $x_{i_1} = 0$.

In general, the $s-1$ degrees of freedom for the k-factor interaction among the i_1-th, i_2-th, ... and i_k-th factors represented by the pencil

$$x_{i_1} + \alpha_{j_2} x_{i_2} + \alpha_{j_3} x_{i_3} + \ldots + \alpha_{j_k} x_{i_k} = \alpha_r \; (j_2, j_3, \ldots j_k \text{ fixed}; \; r = 0, 1, \ldots s-1)$$

correspond to the effect subgroup of order s given by

$$I, A_{i_1} A_{i_2}^{j_2} A_{i_3}^{j_3} \ldots A_{i_k}^{j_k}, A_{i_1}^2, A_{i_2}^{2j_2} \ldots A_{i_k}^{2j_k}, \ldots, A_{i_1}^{(s-1)} A_{i_2}^{(s-1)j_2} \ldots A_{i_k}^{(s-1)j_k},$$

where tj_p $(t = 1, \ldots s-1;\; p = 2, \ldots, k)$ stands for the product of these two numbers in GF (s). The complete orthogonal treatment subgroup of order s^{m-1} is given by the treatment combinations corresponding to the s^{m-1} finite points lying on the $(m-1)$-flat $x_{i_1} + \alpha_{j_2} x_{i_2} + \ldots + \alpha_{j_k} x_{i_k} = 0$. Giving to $j_2, \ldots j_k$ the values $1, 2, \ldots, s-1$, we obtain all the $(s-1)^{k-1}$ pencils corresponding to the k-factor interaction. There are, therefore, $(s-1)^{k-1}$ effect subgroups of order s containing only the symbols $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$, but no others, which correspond to the $(s-1)^k$ degrees of freedom for the interaction $A_{i_1} A_{i_2} \ldots A_{i_k}$.

It would appear from the above that each $(m-2)$-flat in the $(m-1)$-flat at infinity is the vertex of a parallel pencil of s $(m-1)$-flats, which corresponds to an effect subgroup of order s. It is, therefore, appropriate to speak of an effect subgroup of order s as corresponding to an $(m-2)$-flat at infinity. In general, an effect subgroup of order s^k would corres-