ON LATIN AND HYPER-GRAECO-LATIN CUBES AND HYPER-CUBES

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LATIN and Hyper-Graeco-Latin squares were first introduced by Euler[1] in 1782 and have since been extensively studied by a number of mathematicians like Gunther,[2] Cayley,[3] Maillet,[4] Coccoz,[5] Akar,[6] Brocard,[7] Tarry,[8,9,10] Macmahon,[11] MacNeish,[12] Margossian,[13,14] Fisher and Yates,[15,16] Fisher,[17] Bose,[18] Stevens[19] and Norton.[20] To Fisher is due the credit of pointing out their uses in the design of experiments; and with the realisation of their fundamental importance in the theory of this branch of statistics, much attention has been devoted to their study by statisticians in recent years.

2. Fisher introduced the idea of confounding of interactions in symmetrical factorial arrangements, which was extended by Yates to agronomic tests, involving a number of varieties equal to a prime positive integer or a power of prime, in order to increase the 'efficiency' (or accuracy) of the experiments. But it is no disparagement of their work to say that there is a lack of a unified general solution. Nair's work,[21,22] done subsequent to this, was an advance over our then existing knowledge of factorial arrangements in that he developed a method of constructing confounded arrangements in an s^m design, (s a prime positive integer or a power of a prime) in s^2-plot blocks, based on his theory of interchanges derivable from the associated Hyper-Graeco-Latin squares.

But a more complete solution in the case of the general symmetrical factorial arrangement was given by Bose and Kishen,[23] whose investigations achieved the unification and systematization which were lacking in previous work on the subject. Besides succeeding in giving a general method for the formation of confounded arrangements in an s^m design in blocks of s^{m-k} plots and the identification of the confounded degrees of freedom, the authors were able to enunciate the important principle of generalized interaction which enables the best sets of treatment comparisons which may profitably be confounded in any given case to be set down easily and elegantly. It is hoped that the concept of Latin and completely orthogonalized Hyper-Graeco-Latin cubes and hyper-cubes which is now being introduced may be helpful for a fuller understanding of the theory of the general symmetrical factorial arrangement.

3. A Latin cube of the *first order* of side s may be defined as a cube arrangement of s^3 letters, s^2 of each of s kinds, such that each letter occurs exactly s times in each of its three sets of s planes, parallel to the three co-ordinate planes OX_1X_2, OX_1X_3 and OX_2X_3. A Latin cube of the *second order* of side s may be defined as a cube arrangement of s^3 letters, s of each of s^2 kinds, such that each letter occurs exactly once in each of its three sets of s planes parallel to the co-ordinate planes. Thus for $s = 3$, Latin cubes of the first and second orders may be diagrammatically represented as under in Figs. 1 and 2 respectively.

4. If an s-sided Latin cube of the *first order* is superimposed on another s-sided Latin cube of the *first order* such that every letter of one cube occurs exactly s times with every letter of the other cube, the two Latin cubes may be said to be orthogonal to each other. When the letters of the first

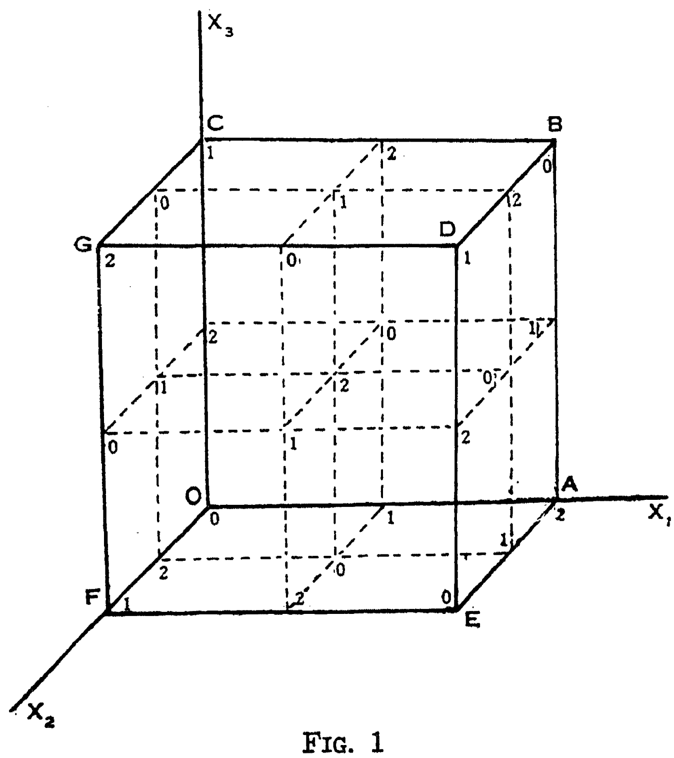

FIG. 1

2×2×2. Latin Cube of First Order

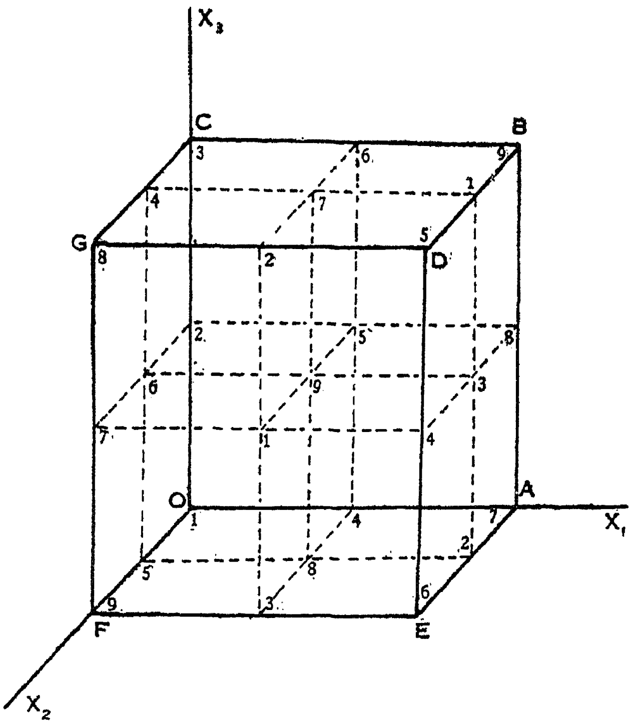

FIG. 2

$2 \times 2 \times 2$. Latin Cube of Second Order

cube are denoted by Latin letters and those of the second cube by Greek letters, and the second is superimposed on the first, the two together may be said to constitute a Graeco-Latin cube of the *first order*. The number of Latin cubes of the first order constituting a completely orthogonalized Hyper-Graeco-Latin cube of the first order has been found to be $s^2 + s - 2$.

5. In general, we may define an s-sided m-fold Latin hyper-cube of the *r-th order* as an m-fold hyper-cube arrangement of s^m letters, s^{m-r} of each of s^r kinds, such that each letter occurs exactly s^{m-r-1} times in each of its m sets of s $(m-1)$-flats, parallel to the m co-ordinate $(m-1)$-flats $OX_1X_2\cdots X_{m-1}$, $OX_1X_2\cdots X_{m-2}X_m$, $\cdots$ $OX_1X_2\cdots X_{i-1}X_{i+1}\cdots X_m$, $\cdots$ $OX_2X_3\cdots X_m$. Two such Latin hyper-cubes, one superimposed on the other, such that every letter of the one occurs exactly s^{m-2r} times with every letter of the other, may be said to be orthogonal to each other. Denoting, as before, letters of the first hyper-cube by Latin letters and those of the second hyper-cube by Greek letters, the composite hyper-cube may be said to constitute an m-fold Graeco-Latin hyper-cube of the *r-th order* and it is obvious that the highest possible value for r is $\frac{m-1}{2}$, when m is odd, and is $\frac{m}{2}$, when m is even.

6. I have been able to establish that Latin cubes and hyper-cubes of the first order of any side exist and that s-sided m-fold Latin hyper-cubes of the rth order $[r \leqq (m-1)$, the sign of equality not holding in certain cases] also exist, s being a prime positive integer or a power of a prime. I have also been able to demonstrate that the existence of an s-sided m-fold Hyper-Graeco-Latin hyper-cube of the *first order* is exactly equivalent to the existence of the finite hyper-dimensional projective geometry $PG(m, s)$, whence it follows that the total number of m-fold Latin hyper-cubes of the *first order* constituting an s-sided m-fold completely orthogonalized Hyper-Graeco-Latin-hyper-cube of the *first order* is $s^{m-1} + s^{m-2} + \cdots\cdots + s^2 + s - (m-1)$. For full details the interested reader is referred to the author's paper on the subject to be published shortly elsewhere.

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