Black hole entropy in string theory – A window into the quantum structure of gravity

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This article is intended as an impressionistic but reasonably self-contained account of black hole entropy, its physical significance, the tortuous historical route to its discovery, how it fits in the framework of string theory, and what we can learn from it about the fundamental degrees of freedom of quantum gravity.

Keywords: Black hole entropy, gravity, string theory.

1. Introduction

ONE of the intriguing properties of a black hole is that it carries entropy much like an ordinary hot body. A beautiful general formula for this entropy due to Bekenstein and Hawking provides a deep connection between quantum mechanics, general relativity, and thermodynamics.

For an ordinary body, its entropy equals the logarithm of the number of ways the atomic constituents of the body configure themselves. But for a black hole, it is far from clear what microscopic constituents might account for its entropy. It has been one of the outstanding open problems in physics to arrive at such a microscopic, statistical understanding of black hole entropy.

There has been considerable progress in recent years in addressing this question in the context of string theory. For a special class of black holes, in many cases, the number of microstates is *exactly* computable and is found to be in precise agreement with the number of states inferred from the entropy to *all orders* in a perturbative expansion. For this comparison to work, it is essential to systematically take into account quantum corrections to the spacetime geometry and the Bekenstein–Hawking formula itself.

Thus, the entropy of a black hole supplies us with precise quantitative information about the fundamental degrees of freedom and offers us glimpses of the inner workings of quantum gravity. These and related developments have led to important insights into the structure of quantum gravity which include in particular the notion of 'holography' and the emerging notion of 'quantum spacetime'.

Apart from its physical significance, the entropy of a black hole makes for a fascinating study in the history of science. It is one of the very rare examples where a scientific idea has gestated and evolved over several decades into an important conceptual and quantitative tool almost entirely on the strength of theoretical considerations. That we can proceed so far with any confidence at all with very little guidance from experiment is indicative of the robustness of the basic tenets of physics. It is therefore worthwhile to place black holes and their entropy in a broader context before coming to the more recent results pertaining to the quantum aspects of black holes within string theory.

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2. The trinity of constants

Perhaps a good measure of the unusual scope and influence of Einstein's ideas is the extent to which his thinking has shaped our understanding of the three fundamental constants of nature – the speed of light c, Planck's constant $\hbar$, and Newton's gravitational constant G. It is also revealing to see the extent to which these constants in turn have circumscribed the development of physics in the last century. In a sense, a very large part of modern physics can be viewed as an elucidation of the meaning of these constants and of the relation between them. With his Special and General Theory of Relativity and with his work in Quantum Theory, Einstein, more than any other single individual, has profoundly transformed the way we think about these constants (It is equally remarkable that this does not exhaust the breadth of Einstein's oeuvre and leaves out his very important work in statistical physics including his work on Brownian motion and critical opalescence.).

Of the three constants, Planck's constant $\hbar$, which governs the laws of the quantum world, has surely had a more pervasive influence on twentieth century physics. Even though Einstein never completely reconciled himself with the full implications of the quantum revolution that unfolded, his own contributions to the subject were nothing but revolutionary. In his paper in the miracle year 1905 on the photo-electric effect, Einstein introduced the light-quantum. With it, he introduced the corpuscular, quantum nature of the electromagnetic waves into physics and opened the door to the particle–wave duality of quantum mechanics. His other contributions to quantum theory were minor perhaps only by his monumental standards since they include the Bose–Einstein statistics and the idea of Bose–Einstein condensation of matter which was verified experimentally only very recently; his work on the specific heat of solids with which began the quantum theory of solids; his work on spontaneous and induced emission of radiation that anticipated quantum electrodynamics and led to the technology of lasers; and his critique of quantum mechanics with the Einstein–Podolsky–Rosen correlations which brought the spooky quantum behaviour into sharp relief.

The second constant, the speed of light c is the cornerstone of the special theory of relativity. The fact that light is an electromagnetic wave travelling at the speed c was a celebrated piece of nineteenth century physics – a consequence of Maxwell's equations for the electromagnetic field. Lorentz and Poincaré, among others, had recognized that Maxwell's equations do not change their form under 'Lorentz transformations' which relate space and time coordinates of observers in uniform motion with respect to each other. This invariance of Maxwell's equations under Lorentz transformations, however, implied that time must dilate and lengths must contract. This was a startling conclusion. For Lorentz and Poincaré, it signified a mysterious new dynamics. It sent them on a wrong track in a fruitless search for some complicated new forces that could explain the contraction of length.

Einstein's great insight was to look for the origin of Lorentz transformations not in the dynamics, which has to do with the forces, but in the *kinematics*, which has to do with the definition of time and length. He recognized that the dilation of time and contraction of space followed from a precise operational definition of 'simultaneity' of events that was purely kinematic. Since it was kinematic, it meant that the invariance under Lorentz transformations and the notion of space and time that it implied must be a property of the laws of motion of all objects and not only of the electro-magnetic field. It is with this crucial observation of Einstein that the speed of light enters into particle mechanics. The relativistic kinematics puts the speed of light as the upper limit for all material propagation, particle or wave, electromagnetic or otherwise. The formula $E = mc^2$, immortalized in popular imagination, then follows from this new kinematics.

The third and the oldest of the trinity of constants, the gravitational constant G, belongs to the general theory of relativity, the greatest of Einstein's achievements. The general theory is actually not about G alone but rather about the two constants G and c together. Newton's law of gravitation requires that the force of gravity acts instantaneously. This is clearly at odds with special relativity which requires that no physical signal can travel faster than the speed of light. For example, according to Newton's gravity, if the sun were to disappear suddenly, its gravity would disappear too and we on earth would come to know about it instantaneously, even though light takes about eight minutes to reach us from the sun. This state of affairs was clearly unsatisfactory and purely for reasons of internal consistency of the physical theory, it was essential to find a broader framework that synthesized Newton's gravity with special relativity. Such a framework would be required for describing phenomena where both G and c are important.

In general relativity, Newton's constant acquires a completely new meaning. For Newton, G is the constant of proportionality that appears in his inverse square law of gravitation. For Einstein, G is the constant that determines the degree to which a given distribution of matter warps space and time. In this new conception, spacetime was no longer a spectator of events but itself a dynamical participant that changed in response to the amount of matter present. It was no longer flat and Euclidean but curved in much the same way as the surface of the earth is round and curved. This curvature of spacetime is, according to Einstein, the origin of gravity. In a flat plane, parallel lines never meet. But in a curved space, as on the surface of the earth, two observers heading straight in two parallel lines starting on the equator will eventually meet at the north pole because of the curvature. In an analogous way, in general relativity, trajectories of two gravitating bodies appear to attract as if because of a force of gravity but it is only because they are moving in a curved spacetime.

The rich harvest of the synthesis effected by general relativity has still not been fully reaped. Just to take two examples of the major efforts in observational astronomy in this century, one is the LIGO project that is seeking to detect the wave of gravitational influence that travels at the speed of light and the other is the WMAP project that has already given us the incredibly detailed picture of the early big-bang cosmology within the framework of general relativity.

One unmistakable pattern in the history of modern physics is the progressive synthesis of ideas by which previously disparate structures are harmonized into a bigger framework. Thus, with special relativity, Einstein harmonized Maxwell's electrodynamics with Newton's mechanics introducing c into mechanics. With general relativity, he further harmonized this structure with Newton's law of gravitation, bringing together c and G.

Clearly one cannot stop here. The next synthesis requires harmonizing the special theory of relativity with quantum mechanics to describe the realm of phenomena where both c and $\hbar$ are important. It is the arena of relativistic quantum field theories that was developed over five decades in which Einstein himself did not play much of a role. Quantum field theory has proved to be the right framework where the duality of wave and particle nature of matter finds its full expression. Starting with Quantum Electrodynamics all the way to the Standard Model of Particle Physics based on quantum gauge theories, quantum field theory has occupied center stage in the study of fundamental interactions. Quantum field theory now encompasses three of the four fundamental interactions including electromagnetic as well as the weak and strong nuclear interactions. We possess a theory of elementary particles, the fundamental blocks of matter and their interactions, that has been tested to great accuracy to distances thousands of times smaller than the atomic nucleus.

That brings us to the final synthesis that still beacons us – a coherent description of physics in the realm where all three fundamental constants are simultaneously important. In other words, a quantum theory of gravity. Gravity, has thus far stubbornly refused to be integrated into the framework of quantum field theory. There is every indication that to do so, another revolutionary change in the paradigm of physics is necessary.

In this search for an overarching framework of quantum gravity that would harmonize quantum mechanics with general relativity, we have had little guidance from experiment. At this historical juncture, there is a peculiar situation in physics. We have two theories that are tremendously successful in their respective domain. Quantum field theory for describing the world at small scales in the realm of elementary particles and General Relativity for describing the world at large scales all the way from our solar system to the universe. There is no experimental compulsion of an unexplained fact that forces us to bring these two theories together. At the same time, at a theoretical level it is absolutely necessary. As they stand, the two theories are in a violent conflict with each other in much the same way that special relativity was at odds with Newton's law of gravitation.

In the best of all possible worlds, theory and experiment work together. Without the sobering guidance from experiment the task of finding the correct theory of quantum gravity is much more difficult and far more risky. And yet history exhorts us to go on. Perhaps, Einstein's struggles towards the general theory of relativity can be our inspiration. Fortunately, we are also given an indirect but definitive piece of information that we can use to peer at the quantum structure of gravity.

It is the entropy of a black hole.

3. Black holes

A black hole is a solution of Einstein's gravitational field equations in the absence of matter that describes the spacetime around a gravitationally collapsed star. Its gravitational pull is so strong that even light cannot escape it.

A black hole is now so much a part of our vocabulary that it can be difficult to appreciate the initial intellectual opposition to the idea of 'gravitational collapse' of a star and of a 'black hole' of nothingness in spacetime by several leading physicists, including Einstein himself.

To quote the relativist Werner Israel: '*There is a curious parallel between the histories of black holes and continental drift. Evidence for both was already non-ignorable by 1916, but both ideas were stopped in their tracks for half a century by a resistance bordering on the irrational*'.

3.1. Schwarzschild and Einstein

On 16 January 1916, barely two months after Einstein had published the final form of his field equations for gravitation[1], he presented a paper to the Prussian Academy on behalf of Karl Schwarzschild[2], who was then fighting a war on the Russian front. Schwarzschild had found a spherically symmetric, static and exact solution of the full nonlinear equations of Einstein without any matter present.

The Schwarzschild solution was immediately accepted as the correct description within general relativity of the gravitational field outside a spherical mass. It would be the correct approximate description of the field around a star such as our sun. But something much more bizarre was implied by the solution. For an object of mass M, the solution appeared to become singular at a radius $R = 2GM/c^2$. For our sun, for example, this radius, now known as the Schwarzschild radius, would be about three kilometers. Now, as long the physical radius of the sun is bigger than three kilometers, the 'Schwarzschild's singularity' is of no concern because inside the sun the Schwarzschild solution is not applicable as there is matter present. But what if the entire mass of the sun was concentrated in a sphere of radius smaller than three kilometers? One would then have to face up to this singularity.

Einstein's reaction to the 'Schwarzschild singularity' was to seek arguments that would make such a singularity inadmissible. Clearly, he believed, a physical theory could not tolerate such singularities. This drove him to write as late as 1939, in a published paper: '*The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality*'.

This conclusion was, however, based on an incorrect argument. Einstein was not alone in this rejection of the unpalatable idea of a total gravitational collapse of a physical system. In the same year, in an astronomy conference in Paris, Eddington, one of the leading astronomers of the time, rubbished the work of Chandrasekhar who had concluded from his study of white dwarfs, a work that was to earn him the Nobel prize later, that a large enough star could collapse.

It is interesting that Einstein's paper on the inadmissibility of the Schwarzschild singularity appeared only two months before Oppenheimer and Snyder published their definitive work on stellar collapse with an abstract that read: '*When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse*'.

Once a sufficiently big star ran out of its nuclear fuel, then there was nothing to stop the inexorable inward pull of gravity. The possibility of stellar collapse meant that a star could be compressed in a region smaller than its Schwarzschild radius and the 'Schwarzschild singularity' could no longer be wished away as Einstein had desired. Indeed it was essential to understand what it means to understand the final state of the star.

3.2. Event horizon

What Einstein referred to as the 'Schwarzschild singularity' is in the matter of fact not a physical singularity at all. It is rather a coordinate singularity because of a bad choice of coordinates. The coordinates that Schwarzschild used to find his solution is more suited for an observer who wants to remain at a fixed distance r from the center. Far away, the constant r surface is time-like, that is, the observer who wants stay fixed at that radius is moving slowly compared to a freely falling observer. But near the Schwarzschild radius, $r = R$, because of the way the space time is curved in the Schwarzschild geometry, the surface $r = R$ is a light-like surface. That is, an observer who wants to remain fixed at that radius has to move at the speed of light. To do so, the observer has to turn on her rockets with infinite acceleration, a physical impossibility. It is this unphysical choice of coordinates that led to the misleading conclusion of a 'singularity' which is not really an intrinsic property of the geometry of spacetime.

Mathematically, a very close analogy for such a coordinate singularity is the singularity in polar coordinates (r, q) in a plane near the origin $r = 0$. The plane is perfectly flat at all points. Its origin is no different from any other point of the plane and the geometry of the plane at the origin is perfectly nonsingular. The proper coordinates at all points for a plane are the cartesian coordinates (x, y). These 'good' coordinates are related to the polar coordinates by $x = r\cos q$ and $y = r\sin q$. Now, at the origin, the polar coordinates are bad because the point $x = 0$, $y = 0$ does not have a unique coordinatization – as long as $r = 0$, all arbitrary values of q would correspond to the same single point $x = 0$, $y = 0$. This coordinate singularity does not signify any intrinsic singularity of the geometry of the plane and in fact can be avoided by simply using the Cartesian coordinates near the origin.

The 'Schwarzschild singularity' can be similarly avoided by a proper choice of 'good' coordinates. In general relativistic spacetime, the analog of a Cartesian coordinate frame is the coordinate frame of an observer who is freely falling through spacetime with her rocket engines switched off.

The surface $r = R$, even though not singular and perfectly ordinary in terms of its local geometry, is nevertheless rather peculiar in terms of the global causal structure. Since the surface is moving at the speed of light, once an observer crosses it, she cannot come out no matter how powerful her rockets. Because to do so, she would have to move faster than the speed of light. Thus, the $r = R$ surface is the boundary of the 'inside of a black hole' from behind which even light cannot escape to the observer who is sitting far away from the black hole. This boundary is then in the causal sense a one-way surface. From outside, we can send signals across the surface but can never receive signals coming out from it. Such a one-way surface is called an 'Event horizon'. The black hole is more precisely then the region of spacetime bounded by the event horizon. It is literally a hole in spacetime which is black because no light can come out of it. The name 'black hole' for this final state of the collapsed star, a spacetime with an event horizon, was proposed by John Wheeler in 1967 and it stuck.

Much of the interesting physics of a black hole, both classical and quantum, and the fact that a black hole has entropy, has to do with the existence of an event horizon.

3.3. Simple and yet complex

A black hole is at once the most simple and the most complex object.

It is the most simple in that it is completely specified by its mass, spin, and charge. This remarkable fact is a consequence of the so-called 'No Hair Theorem'. For an astrophysical object like the earth, the gravitational field around it depends not only on its mass but also on how the mass is distributed and on the details of the oblateness of the earth and on the shapes of the valleys and mountains. Not so for a black hole. Once a star collapses to form a black hole, the gravitational field around it forgets all details about the star that disappears behind the even horizon except for its mass, spin, and charge. In this respect, a black hole is very much like a structure-less elementary particle such as an electron.

And yet it is the most complex in that it possesses a huge entropy. In fact the entropy of a solar mass black hole is enormously bigger than the thermal entropy of the star that might have collapsed to form it. As we will see in §4, entropy gives an account of the number of microscopic states of a system. Hence, the entropy of a black hole signifies an incredibly complex microstructure. In this respect, a black hole is very unlike an elementary particle.

Understanding the simplicity of a black hole falls in the realm of classical gravity. By the early seventies, full fifty years after Schwarzschild, a reasonably complete understanding of gravitational collapse and of the properties of an event horizon was achieved within classical general relativity. The final formulation began with the singularity theorems of Penrose, area theorems of Hawking and culminated in the laws of black hole mechanics which we will come to in §5.

Understanding the complex microstructure of a black hole implied by its entropy falls in the realm of quantum gravity. To understand the meaning of the entropy of a black hole and its implications, let us first recall what we understand by entropy in thermodynamics and statistical physics.

4. Entropy and microstates

Entropy is among the more subtle concepts in physics. It is not a property of a single microstate like energy or charge, but gives instead a count of the *total* number of microscopic states available to a macroscopic system that has fixed total energy and total charge.

Entropic and statistical considerations have been used to great advantage in physics to draw profound conclusions about the atomic microstructure from gross thermodynamic properties such as temperature and heat exchange. For example, already in the 19th century, some of the far-seeing physicists of the time were keenly aware of the crisis of classical physics based purely on statistical considerations.

For example, by looking at the specific heat of gases such as oxygen, Maxwell and Jeans had correctly concluded that classical molecular theory of gases was in serious trouble. The classical degrees of freedom of the theory implied much too large thermodynamic entropy. Similarly, Gibbs had inferred the strict quantum indistinguishability of oxygen molecules from considerations of thermodynamics and statistics. It is remarkable that these conclusions could be drawn at a time when full-fledged quantum mechanics was still several decades in the future. It is all the more remarkable that they were not based on subtle experiments as one might expect for a theory dealing with the atomic structure. Rather it was the logic of these enquiries which was subtly directed at explaining some gross thermodynamic feature of everyday gases such as their entropy and specific heat.

These are useful historical analogies to keep in mind as we look at the road ahead for quantum gravity. We dwell on these analogies a bit in this section to gain a precise understanding of the relation between entropy and state-counting so that we can better appreciate the physical significance of the entropy of a black hole.

4.1. *Irreversibility and entropy*

Heat flows from a hot body to a cold body but not the other way around. How can we quantify this irreversibility of everyday experience?

The answer to this question came, among others, from a French engineer, Sadi Carnot who wanted to know how to build the most efficient steam engine to extract maximum possible work from it. He concluded that the most efficient engine is a reversible one, a thermodynamic analog of a frictionless engine. The efforts to quantify the notion of reversibility led to the notion of entropy. Define a quantity called entropy S as follows. If you add heat ΔQ to a body at temperature T, then the change in the entropy ΔS is given by

$$\Delta S = \frac{\Delta Q}{T}. \quad (1)$$

The important property of entropy is that in a reversible process, the total change in entropy is zero. Entropy is then an intrinsic property of a given system and is a function of the energy and the volume of the system. This allowed Carnot to enunciate the 'Second Law of Thermodynamics' which states that in an irreversible process entropy always increases.

The second law of thermodynamics explains the irreversibility of heat flow as follows. If heat $|\Delta Q|$ flows from a body at temperature T_1 to a body at temperature T_2, then the second body gains in entropy and the first body loses in entropy. The net change in entropy is then

$$\Delta S = |\Delta Q| \left(\frac{1}{T_2} - \frac{1}{T_1} \right). \quad (2)$$

Since the second law requires $\Delta S > 0$ for an irreversible process, it implies that heat can flow only if $T_1 > T_2$.

At this stage, the second law is a phenomenological law. A microscopic understanding of the second law was completed by Boltzmann who gave a statistical interpretation of entropy.

4.2. *Entropy and disorder*

Boltzmann related the thermodynamic entropy S of a system to the total number Ω of different ways the microscopic constituents of the system can arrange themselves. He gave the fundamental relation

$$S = k \log \Omega, \quad (3)$$

where k is Boltzmann's constant.

Boltzmann's relation explains the second law of thermodynamics and the associated irreversibility from a microscopic point of view as a statistical tendency towards disorder. If you shake a jigsaw puzzle it is more likely to break than assemble itself simply because there are hugely more states when its broken than when it is not. In other words, the system has more entropy when it is broken than when it is not. If we shake a puzzle in a box and look at it, statistically it is much more likely to be found in a broken state than in the assembled state. This explains why entropy always increases but only in a statistical sense.

To understand better the relation between thermodynamics and microscopic degrees of freedom, consider the entropy of oxygen in a room from this point of view. The thermodynamic entropy can be measured easily. To check Boltzmann's relation, we need to know Ω, the total number of ways for distributing N molecules of oxygen in a room of volume V. Now, a quantum particle like oxygen has wavelike nature and has a characteristic wavelength l called its thermal de Broglie wavelength which can be thought of as its characteristic size. Since each molecule occupies volume l^3, one can imagine that the room is divided into boxes, $\frac{V}{l^3}$, in number. There are $\frac{V}{l^3}$ ways a single molecule can be distributed in these boxes in the room. If we have N molecules, the total number would then be given by

$$\Omega = \frac{1}{N!}\left(\frac{V}{l^3}\right)^N . \tag{4}$$

Here the crucial factor $N!$ is included because all oxygen molecules are identical. With l the thermal de Broglie wavelength, the logarithm of this quantity gives the correct answer for the thermodynamic entropy of a dilute gas like oxygen. This simple calculation was one of the great successes of molecular theory of gases in the nineteenth century which explained a gross, thermodynamic property in terms of a microscopic counting. It was in a sense a first peek at the atomic structure of matter.

4.3. *Quantum counting and classical over-counting*

There are a number of features of eq. (4) that are worth noting because they reveal important aspects of the concept of entropy and its physical significance. With Boltzmann's relation connecting this counting with the entropy it already contains important hints about the quantum structure of matter.

First, without the factor $N!$, the Boltzmann relation would not be satisfied. This fact, first deduced by Gibbs from purely thermodynamic reasoning, attests to the strict indistinguishability of quantum oxygen molecules. Full theoretical understanding was this fact would require among other things, Bose–Einstein statistics and the spin-statistics theorem that came much later.

Second, in a strictly classical theory, in the limit $\hbar \to 0$, the thermal de Broglie wavelength would be zero. A classical point particle does not really occupy any space at all. As a result, the classical counting would give an infinity of states, and $k \log\Omega$ would be infinite even though physical entropy of a gas is finite. Finiteness of entropy is thus an indication of the quantum nature of the degrees of freedom.

This classical over-counting of degrees of freedom is typical and we will encounter it in the context of black holes as well. It manifests itself even in other statistical quantities. For example, the classical specific heat of oxygen is too large compared to the experimental value. This again is a consequence of a more subtle over-counting.
Thinking of this problem, Maxwell remarked in a lecture given in 1875: '*Every additional degree of complexity which we attribute to the molecule can only increase the difficulty of reconciling the observed with the calculated value of the specific heat. I have now put before you what I consider the greatest difficulty yet encountered by the molecular theory*'[3].

Maxwell's difficulty had to do with the failure of classical equipartition theorem which assigns equal energy to all degrees of freedom. It was a thermodynamic manifestation of the inadequacy of classical ideas. Pondering over the same difficulty, Jeans made a prescient remark in 1890 that somehow 'the degrees of freedom seem to be frozen'.

In the full quantum theory which was to emerge several decades later, the resolution comes indeed from the fact at low temperature, average thermal energy would be much smaller than the quantum of energy needed to excite a degree of freedom such as a vibration of a molecule. In this case, such a degree of freedom is effectively frozen out as foreseen by Jeans. As a result, the classical equipartition theorem that Maxwell was using is not applicable thereby avoiding the conflict of theory with observation.

We are drawing this historical analogy to underscore the point that even when the full picture about the quantum theory of matter was very far from clear, it was possible to learn a great deal about the shape of the theory to come from this kind of thermodynamic and statistical considerations.

The situation with regard to quantum gravity is not quite the same but is in some ways analogous. Statistical reasoning has proved to be a valuable guide also in understanding the physics of black holes. One hopes that whatever may be the final form that the theory of quantum gravity takes, the insights that we can glean from the entropy of a black hole will be a part of it.

5. Black hole entropy

Before coming to the statistical aspects, let us first understand the thermodynamic aspects of a black hole.

5.1. *Bekenstein*

Jabob Bekenstein, then a graduate student of Wheeler, asked a simple-minded but incisive question[4]. What happens if you throw a bucket of hot water into a black hole? The entropy of the world outside the black hole would then decrease and the second law of thermodynamics would be violated. Should we give up this law that was won after half a century of hard struggle now in the presence of black holes?

Since the inside of the event horizon is never accessible by causal process to outside observers, whatever falls in it is forever lost. This fortunately does not affect the usual conservation laws of quantities such as energy and charge. For example, the energy of the bucket would be lost to the outside world but the energy or equivalently the mass of the black hole will go up by the same amount. The mass of the black hole can be measured from outside from its gravitational pull so if we keep track of the energy content of a black hole in our accounting of energy, then energy would continue to be conserved.

This suggested that even for entropy, if one could somehow associate an entropy with a black hole, then the

Table 1. Black hole thermodynamics

Laws of thermodynamics	Laws of black hole mechanics
Temperature is constant throughout a body at equilibrium, T = constant	Surface gravity is constant on the event horizon, k = constant
Energy is conserved, dE = TdS.	Energy is conserved, d$M = \frac{k}{8p}$ dA.
Entropy never decrease, $\Delta S \geq 0$	Area never decreases, $\Delta A \geq 0$

A, area of the horizon; M, mass of the black hole; and k, surface gravity which can be thought of roughly as the acceleration at the horizon (We have stated these laws for black holes without spin and charge but more general form is known.)

second law of thermodynamics could be saved if we also keep track of the entropy of a black hole in our accounting of total entropy. But the 'No-Hair' theorem mentioned earlier showed that there were no other attributes of the black hole apart from its mass, charge, and spin that could be measured from outside.

There is one quantity however, Bekenstein noted, namely the area of the black hole which behaved like entropy in many ways. For the Schwarzschild black hole, this is simply the area of the event horizon which equals $4\pi R^2$ where R is the Schwarzschild radius. For Bekenstein, the analogy was suggested by the remarkable laws of black hole mechanics, crystallized by Bardeen, Carter, and Hawking, which had a striking resemblance with the three laws of thermodynamics for a body in thermal equilibrium (Table 1).

5.2. *Hawking radiation*

This analogy of Bekenstein was not immediately accepted because there was a serious difficulty with it. If a black hole has entropy and energy then it must also have temperature as can be seen from the definition of entropy (see §4.1). Now, any hot body must radiate and so also must a black hole with temperature. This conclusion was preposterous from the point of classical general relativity since after all a black hole was so named because it was perfectly black and nothing could come out of it.

Initially, Hawking among others, was willing to give up the second law in the face of this difficulty. Very soon though, he realized in his classic paper that a black hole could indeed have temperature once you include quantum effects[5]. In a quantum theory, virtual particles and antiparticles are constantly being created and annihilated from vacuum. Usually, they cannot be separated into real particles without violating conservation of energy because that would amount to creating a particle–antiparticle pair out of nothing. Near the event horizon, however, the antiparticle can fall into the black hole and the particle can escape to infinity as a real particle. Energy can be conserved in the process because the mass of black hole reduces accordingly. Hawking showed that the spectrum of these particles radiated from the black hole is exactly as if they are being radiated by a hot body at temperature T.

This temperature T of the black hole, now known as the Hawking temperature, is given by a simple formula

$$T = \frac{\hbar k}{2\pi}, \tag{5}$$

where k is the surface gravity encountered earlier. With this remarkable discovery, the table above becomes more than just an analogy. Indeed the left column is now precisely the same as the right column with the identification

$$S = \frac{Ac^3}{4\hbar G} = \frac{A}{4l^2}, \tag{6}$$

Here the length l is the Planck length 10^{-33} cm, a fundamental length constructed from the trinity of constants $l^2 = G\hbar/c^3$. It is remarkably general formula valid in all dimensions and for all kinds of black holes with mass, charge, and spin.

Note that in the classical limit $\hbar \to 0$, the temperature vanishes, as it should since a black hole is really black classically. More importantly, in this limit entropy would become infinite. This is exactly as what we saw for oxygen gas in §4 and is the usual problem of classical over-counting. The finite quantum entropy of a black hole therefore signifies a certain discreteness of the degrees of freedom. This entropy is at present the only known physical quantity that involves all three fundamental constants of nature. It is therefore a precious clue about the microscopic structure of quantum gravity.

The discovery of thermodynamic entropy of a black hole in this way resolves the puzzle of Bekenstein about the apparent violation of the second law of thermodynamics. But it raises an even more interesting puzzle. Since the entropy of the black hole behaves in every respect like any other entropy that one encounters in statistical mechanics, what are the microstates of the black hole that can account for this thermodynamic entropy?

This remained an open problem for over two decades after Hawking's discovery. A complete understanding of the entropy of general black holes is still lacking, but there has been remarkable progress in addressing this question within the framework of string theory.

6. String theory and black holes

Let us recall a few relevant facts about string theory (for more details about string theory see the articles by David Gross and Ashoke Sen in this special section.) which is presently the leading candidate for a quantum theory of gravity.

String theory posits that the fundamental degrees of freedom are string-like extended objects instead of point-like elementary particles as assumed in quantum field theory. Different elementary particles arise as different oscillation modes of this fundamental string. Finding the spectrum of a fundamental string is then analogous to finding what frequencies of sound will be produced by a sitar string. In this analogy, each musical note produced by the string would correspond to an elementary particle. One of the early surprises in the investigations of quantum string theories was that the spectrum of string theory always contained the graviton – the elementary particle that corresponds to a gravitational wave rippling through spacetime which carries the force of gravity. This striking fact was a natural consequence of the theory and was not put in by hand. Thus string theory is automatically a quantum theory of gravity. In a sense, quantum gravity is not only possible within string theory but is in fact necessary. Furthermore, when the gravitational coupling G is small, the interactions of the gravitons within string theory are free of the unphysical infinities that plagued earlier attempts to formulate quantum gravity within the framework of quantum field theory.

Earlier developments in string theory were limited to situations where gravitational interactions are weak. In the context of black holes however, the gravitational interactions are strong enough to warp spacetime into a black hole. Black holes therefore obtain a useful laboratory for testing the formalism of string theory beyond weak coupling. One of the striking successes of string theory is that for a special class of black holes, one can indeed explain the thermodynamic Bekenstein–Hawking entropy in terms of underlying microstates which can be counted exactly. What is more, in some examples, one can compute the corrections to the Bekenstein–Hawking entropy systematically to all orders in a perturbative expansion in inverse area and these too agree precisely with the microscopic counting.

The beautiful agreement that we find between the microscopic counting and the macroscopic, thermodynamic entropy not only resolves a long-standing puzzle raised about the interpretation of black hole entropy but also gives a strong hint that string theory provides a consistent framework for quantum gravity even at strong coupling.

6.1. *Counting black holes*

Supersymmetry is a generalization of Lorentz transformations. Just as special relativity requires that the laws of physics be invariant under Lorentz transformations, string theory requires that the laws of physics be invariant under a bigger symmetry, supersymmetry. Combining supersymmetry with Einstein's theory of gravity leads to a generalization of general relativity called supergravity.

The chief tool in dealing with the entropy of black holes in string theory is the spectrum of 'supersymmetric states'. Supersymmetric states of a theory are a special class of states that carry both mass and charge and have the property that their spectrum does not change as one changes the coupling constant of the theory. As a result, the number of such states can be counted reliably when Newton's constant G is small and gravitational interactions are weak. The counting is much easier in this limit as we will see below. Now, if one increases the value of G, then gravity becomes important, and a state with mass M and charges Q_1, Q_2, ... undergoes gravitational collapse. Since it is a supersymmetric state, the number of states at large G implied by the entropy of the corresponding black hole must equal the number of states counted at small G.

The most well-studied example that gives a microscopic account of the thermodynamic entropy is in five spacetime dimensions with three kinds of charges: Q_1, Q_2, Q_3 (ref. 6). In this case, it is possible to count the number of supersymmetric states with these charges and in the limit of large charges, number of such states $\Omega(Q_1, Q_2, Q_3)$ grows exponentially in a way that matches precisely with thermodynamic entropy $S(Q_1, Q_2, Q_3) \equiv \frac{A}{4l^2}$, of black holes with the same charges,

$$S(Q_1, Q_2, Q_3) = k \log \Omega(Q_1, Q_2, Q_3) = 2\pi k \sqrt{Q_1 Q_2 Q_3}. \quad (7)$$

6.2. *Black holes as strings*

To describe how this comparison is carried out, we consider instead a simpler system in four spacetime dimensions that has only two charges p and q. One advantage of this system is that the microscopic counting can be done more easily and exactly even for small charges. As a result, a much more detailed comparison can be carried out including all order corrections to the Bekenstein–Hawking formula.

String theory naturally lives in nine space dimensions. To obtain the physical space of three dimensions, it is necessary to 'compactify' or to curl up the extra six dimensions into small internal space. Thus, in string theory one imagines that at each point in physical space there is attached a small ball of six dimensions. Now suppose that one of the directions of the internal space is a circle. Consider a string wrapping q times with momentum p along this circle. The string looks point-like in four dimensions. Usually along a string extending vertically, the oscillations can either move up or move down. In the type of string theory used in this context called the 'heterotic' string, if we have only up-moving oscillations with total energy $N = pq$, then this state is supersymmetric.

Now, a string extending along one of the nine spatial directions has eight transverse directions. In addition, in heterotic string theory, there are sixteen internal dimensions along which the string can carry up-moving oscillations. These extra internal dimensions have to do with the fact that in heterotic string, states can carry sixteen kinds of charges. Therefore, altogether, we have twenty-four up-moving oscillations. Each oscillation has frequency labelled by an integer, $n = 1, 2, 3, ..., \infty$ which basically counts the number of wavelengths of the oscillations that can fit on the circle travelling around the circumference. We need to distribute the total energy N among all these oscillators and find out how many ways there are of doing it to find the total number of states with charges p and q. This problem then maps to a well-known class of problems analysed by Hardy and Ramanujan. The total number of our black hole states then equals the number of ways one can partition an integer N into a sum of integers, using integers of 24 different colours. This quantity is usually denoted as $p_{24}(N)$. For example, the total number of ways of partitioning the integer 5 using integers of only one colour would be denoted $p_1(5)$. It is easy to find this number, since

$$\begin{aligned} 5 &= 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 2 + 2 + 1 \\ &= 3 + 1 + 1 = 3 + 2 = 4 + 1 = 5, \end{aligned}$$

and hence $p_1(5) = 7$. It is also easy to see that this number grows very rapidly, in fact exponentially, as we increase either the integer or the number of colours at our disposal.

To count our black holes with large charges we then need to find $p_{24}(N)$ for large values of N. The answer can be found exactly to all orders,

$$\begin{aligned} \log \Omega(p, q) &\sim 4\pi\sqrt{pq} - \frac{27}{2}\log\sqrt{pq} - \log\sqrt{2} \\ &\quad - \frac{675}{32p\sqrt{pq}} - \frac{675\times 9}{2048 p^2 pq} - \ldots \end{aligned} \tag{8}$$

How does this microscopic counting compare with the macroscopic entropy?

6.3 *Beyond Bekenstein and Hawking*

In this particular example considered above, it turns out, on the macroscopic side, classical spacetime is singular and the entropy vanishes. This seems to be in flat contradiction with the result from the microscopic counting. However, things get more interesting because string theory implies calculable corrections to general relativity. Einstein's equations are nonlinear partial differential equations that involve only two derivatives of the dynamical fields. From a modern perspective, these equations are expected to be just the low energy approximation and the full equations are expected to contain terms with higher derivatives of the dynamical fields coming from various quantum corrections. In the presence of the higher derivative terms, both the solution and the Bekenstein–Hawking formula itself get modified. There exists an elegant generalization of the Bekenstein-Hawking formula due to Wald[7,8] that correctly incorporates the effects of the higher derivative terms. It gives the entropy as an infinite series

$$S = a_0 A(Q) + a_1 \log A(Q) + \frac{a_2}{A(Q)} + \ldots, \tag{9}$$

where the coefficients a_i can be computed explicitly from the specific form of the generalization of Einstein's equations that follows from string theory.

To apply Wald's formula one must first find the quantum corrections to Einstein's equations and then find the generalization of the Schwarzschild solution including these corrections. It would appear like an impossible task to solve these highly nonlinear, higher derivative partial differential equations. Fortunately, it turns out that using various available techniques and supersymmetry[9–12], it is possible to compute the higher derivative terms with precise numerical coefficients in string theory and find the corrected solution. One then finds that the solution including these quantum corrections has finite area[13,14] and is given by $A = 8\pi l^2 \sqrt{pq}$ in terms of the charges p, q above and Planck length l. Using the Bekenstein–Wald entropy formula one then finds that the perturbative expansion in inverse area in eq. (9) is the same as the expansion for large charges in $1/\sqrt{pq}$ in eq. (9). The coefficients a_i can be computed exactly and are such that the entropy S in formula above match precisely with the infinite expansion of the microscopic counting (eq. (9)) except for an additive constant that cannot yet be determined[13,15].

7. Glimpses of quantum gravity

String theory is at present the only known framework for understanding black hole entropy in terms of counting albeit only in some special cases. The fact that even the corrections to the entropy can be understood in terms of microscopic counting to all orders is encouraging. It remains a challenge to see how these results can be extended to the Schwarzschild black hole without using the crutches of supersymmetry.

There are other important insights that have emerged from the study of black holes, most notably, the notion of 'Holography'. Bekenstein noted that the total number of degrees of freedom in a region must be proportional to the area of the region (and not its volume as one might naively expect) measured in units of the Planck length. Otherwise, black hole formation in this region would violate the second law of thermodynamics. This observation implies a dramatic reduction in the number of degrees of freedom of quantum gravity. It will take us too far afield to discuss these developments relating to holography in any detail here.

It is not clear yet what form the final formulation of quantum gravity will take but there is every indication that string theory will be a part of it. In the absence of direct experimental evidence, one can subject the formalism of string theory to stringent tests of consistency. The striking agreement between thermodynamic, macroscopic properties of black holes and the microscopic structure of the theory assures us that we might be on the right track.

What would Einstein have thought of this road to quantum gravity? In his own research, he was a master of statistical reasoning and used it with incomparable skill to establish the quantum reality of atoms and light. He was also the one who gave us the theory of gravity based on the geometry of spacetime. Perhaps he would have appreciated the current struggles to learn about quantum gravity from the interplay between geometry and thermodynamics.

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Max Planck and Albert Einstein. Photo courtesy: AIP Emilio Segre Archives.