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On a problem of Pillai with Fibonacci numbers and powers of 2

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Abstract. In this paper, we find all integers c having at least two representations as a difference between a Fibonacci number and a power of 2.

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Its first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \ldots$$

In 1936, Pillai [6, 7] showed that if a and b are coprime then there exists $c_0(a, b)$ such that if $c > c_0(a, b)$ is an integer, then the equation $c = a^x - b^y$ has at most one positive integer solution (x, y). In the special case $(a, b) = (3, 2)$ which was studied before Pillai by Herschfeld [3, 4], Pillai conjectured that the only integers c admitting two representations of the form $3^x - 2^y$ are given by

$$1 = 3-2 = 3^2-2^3, \quad -5 = 3-2^3 = 3^3-2^5, \quad -13 = 3-2^4 = 3^5-2^8.$$

This was confirmed by Stroeker and Tijdeman [8] in 1982. Here we study a related problem and find all positive integers c admitting two representations of the form $F_n - 2^m$ for some positive integers n and m. We assume that representations with $n \in \{1, 2\}$ (for which $F_1 = F_2$) count as one representation just to avoid trivial 'parametric families' such as $1 - 2^m = F_1 - 2^m = F_2 - 2^m$, and so we always assume that $n \geq 2$. Notice the solutions

$$\begin{aligned} 1 &= 5-4 = 3-2(= F_5 - 2^2 = F_4 - 2^1), \\ -1 &= 3-4 = 1-2(= F_2 - 2^2 = F_2 - 2^1), \\ -3 &= 5-8 = 1-4 = 13-16(= F_5 - 2^3 = F_2 - 2^2 = F_7 - 2^4), \\ 5 &= 21-16 = 13-8(= F_8 - 2^4 = F_7 - 2^3), \\ 0 &= 8-8 = 2-2(= F_6 - 2^3 = F_3 - 2^1), \\ -11 &= 21-32 = 5-16(= F_8 - 2^5 = F_5 - 2^4), \\ -30 &= 34-64 = 2-32(= F_9 - 2^6 = F_3 - 2^5) \\ 85 &= 4181-4096 = 89-4(= F_{19} - 2^{12} = F_{11} - 2^2). \end{aligned} \tag{1}$$

We prove the following theorem.

Theorem 1. *The only integers c having at least two representations of the form $F_n - 2^m$ are $c \in \{0, 1, -1, -3, 5, -11, -30, 85\}$. Furthermore, for each c in the above set, all its representations of the form $F_n - 2^m$ with integers $n \geq 2$ and $m \geq 1$ appear in the list* (1).

2. A lower bound for a linear forms in logarithms of algebraic numbers

In this section, we state a result concerning lower bounds for linear forms in logarithms of algebraic numbers which will be used in the proof of our theorem.

Let η be an algebraic number of degree d whose minimal polynomial over the integers is

$$g(x) = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}).$$

The logarithmic height of η is defined as

$$h(\eta) = \frac{1}{d}\left(\log |a_0| + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

Let $\mathbb{L}$ be an algebraic number field and $d_{\mathbb{L}}$ be the degree of the field $\mathbb{L}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|, 3\},$$

and put

$$\Lambda = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l.$$

The following result is due to Matveev [5].

Theorem 2. *If $\Lambda \neq 0$ and $\mathbb{L} \subset \mathbb{R}$, then*

$$\log|\Lambda| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

3. Proof of Theorem 1

Assume that $(n, m) \neq (n_1, m_1)$ are such that

$$F_n - 2^m = F_{n_1} - 2^{m_1}.$$

If $m = m_1$, then $F_n = F_{n_1}$ and since $\min\{n, n_1\} \geq 2$, we get that $n = n_1 = 2$, so $(n, m) = (n_1, m_1)$, which is not the case. Thus, $m \neq m_1$, and we may assume that $m > m_1$. Since

$$F_n - F_{n_1} = 2^m - 2^{m_1}, \tag{2}$$

and the right–hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \geq 3$ and $n_1 \geq 2$. We use the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad \text{for all } k \geq 0,$$

where $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. It is well-known that

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \quad \text{for all } k \geq 1.$$

In (2), we have

$$\begin{aligned} \alpha^{n-4} &\leq F_{n-2} \leq F_n - F_{n_1} = 2^m - 2^{m_1} < 2^m, \\ \alpha^{n-1} &\geq F_n > F_n - F_{n_1} = 2^m - 2^{m_1} \geq 2^{m-1}, \end{aligned} \tag{3}$$

therefore

$$n-4 < c_1 m \text{ and } n-1 > c_1(m-1), \text{ where } c_1 = \log 2/\log\alpha = 1.4402\ldots. \tag{4}$$

If $n < 400$, then $m < 300$. We ran a computer program for $2 \leq n_1 < n \leq 400$ and $1 \leq m_1 < m < 300$ and found only the solutions from list (1). From now on, $n \geq 400$. By the above inequality (4), we get that $n > m$. Thus, we get

$$\begin{aligned} \left|\frac{\alpha^n}{\sqrt{5}} - 2^m\right| &= \left|\frac{\beta^n}{\sqrt{5}} + \frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} - 2^{m_1}\right| \leq \frac{\alpha^{n_1} + 2}{\sqrt{5}} + 2^{m_1} \\ &\leq \frac{2\alpha^{n_1}}{\sqrt{5}} + 2^{m_1} < 2\max\{\alpha^{n_1}, 2^{m_1}\}. \end{aligned}$$

Dividing by 2^m, we get

$$\left|\sqrt{5}^{-1}\alpha^n 2^{-m} - 1\right| < 2\max\left\{\frac{\alpha^{n_1}}{2^m}, 2^{m_1-m}\right\} < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}, \tag{5}$$

where for the last right-most inequality we used (3) and the fact that $2 < \alpha^2$. For the left-hand side, we use Theorem 2 with the data

$$l = 3,\ \eta_1 = \sqrt{5},\ \eta_2 = \alpha,\ \eta_3 = 2,\ d_1 = -1,\ d_2 = n,\ d_3 = -m.$$

We take $\mathbb{L} = \mathbb{Q}(\sqrt{5})$ for which $d_{\mathbb{L}} = 2$. Then we can take $A_1 = 2h(\eta_1) = \log 5$, $A_2 = 2h(\eta_2) = \log\alpha,\ A_3 = 2h(\eta_3) = 2\log 2$. We take $D = n$. We have

$$\Lambda = \sqrt{5}^{-1}\alpha^n 2^{-m} - 1.$$

Clearly, $\Lambda \neq 0$, for if $\Lambda = 0$, then $\alpha^{2n} \in \mathbb{Q}$, which is false. The left-hand side of (6) is bounded by Theorem 2, as

$$\log|\Lambda| > -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log n)(\log 5)(2\log\alpha)(2\log 2).$$

Comparing with (5), we get

$$\min\{(n-n_1-6)\log\alpha, (m-m_1-1)\log 2\} < 1.1\times 10^{12}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} < 1.2\times 10^{12}(1+\log n).$$

Now the argument splits into two cases.

Case 1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (n-n_1)\log\alpha$. In this case, we rewrite (2) as

$$\left|\frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1} - 2^m\right| = \left|\frac{\beta^n - \beta^{n_1}}{\sqrt{5}} - 2^{m_1}\right| < 2^{m_1} + 1 \le 2^{m_1+1},$$

so

$$\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}2^{-m} - 1\right| < 2^{m_1-m-1}. \tag{6}$$

Case 2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (m-m_1)\log 2$. In this case, we rewrite (2) as

$$\left|\frac{\alpha^n}{\sqrt{5}} - 2^{m_1}(2^{m-m_1}-1)\right| = \left|\frac{\beta^n + \alpha^{n_1} - \beta^{n_1}}{\sqrt{5}}\right| < \frac{\alpha^{n_1}+2}{\sqrt{5}} < \alpha^{n_1},$$

so

$$\left|(\sqrt{5}(2^{m-m_1}-1))^{-1}\alpha^n 2^{-m_1} - 1\right| < \frac{\alpha^{n_1}}{2^m - 2^{m_1}} \le \frac{2\alpha^{n_1}}{2^m} \le 2\alpha^{n_1-n+4} < \alpha^{n_1-n+6}. \tag{7}$$

Inequalities (6) and (7) suggest studying lower bounds for the absolute values of

$$\Lambda_1 = \left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}2^{-m} - 1 \quad \text{and} \quad \Lambda_2 = (\sqrt{5}(2^{m-m_1}-1))^{-1}\alpha^n 2^{-m_1} - 1.$$

We apply Theorem 2 again. We take, in both cases, $l = 3$, $\eta_2 = \alpha$, $\eta_3 = 2$. For Λ_1, we have $d_2 = n_1,\ d_3 = -m$, while for Λ_2 we have $d_2 = n,\ d_3 = -m_1$. In the both cases we take $D = n$. We take

$$\eta_1 = \frac{\alpha^{n-n_1} - 1}{\sqrt{5}} \quad \text{or} \quad \eta_1 = \sqrt{5}(2^{m-m_1} - 1),$$

according to whether we work with Λ_1 or Λ_2, respectively. For Λ_1 we have $d_1 = 1$ and for Λ_2 we have $d_1 = -1$. In both cases $\mathbb{L} = \mathbb{Q}(\sqrt{5})$ for which $d_{\mathbb{L}} = 2$. The minimal polynomial of η_1 divides

$$5X^2 - 5F_{n-n_1}X - ((-1)^{n-n_1} + 1 - L_{n-n_1}) \quad \text{or} \quad X^2 - 5(2^{m-m_1} - 1)^2,$$

respectively, where $\{L_k\}_{k\ge 0}$ is the Lucas companion sequence of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$, $L_{k+2} = L_{k+1} + L_k$ for $k \ge 0$ for which its Binet formula of the general term is

$$L_k = \alpha^k + \beta^k \qquad \text{for all } k \ge 0.$$

Thus,

$$h(\eta_1) \le \frac{1}{2}\left(\log 5 + \log\left(\frac{\alpha^{n-n_1} + 1}{\sqrt{5}}\right)\right) \quad \text{or} \quad \log(\sqrt{5}(2^{m-m_1} - 1), \tag{8}$$

respectively. In the first case,

$$h(\eta_1) < \frac{1}{2}\log(2\sqrt{5}\alpha^{n-n_1}) < \frac{1}{2}(n-n_1+4)\log\alpha < 7\times 10^{11}(1+\log n), \tag{9}$$

and in the second case

$$h(\eta_1) < \log(8 \times 2^{m-m_1}) = (m - m_1 + 3)\log 2 < 1.3 \times 10^{12}(1 + \log n).$$

So, in both cases, we can take $A_1 = 2.6 \times 10^{12}(1+\log n)$. We have to justify that $\Lambda_i \ne 0$ for $i = 1, 2$. But $\Lambda_1 = 0$ means

$$(\alpha^{n-n_1} - 1)\alpha^{n_1} = \sqrt{5} \times 2^m.$$

Conjugating this relation in $\mathbb{Q}$, we get that

$$(\alpha^{n-n_1} - 1)\alpha^{n_1} = -(\beta^{n-n_1} - 1)\beta^{n_1}. \tag{10}$$

The absolute value of the left-hand side is at least $\alpha^n - \alpha^{n_1} \ge \alpha^{n-2} \ge \alpha^{398}$, while the absolute value of the right–hand side is at most $(|\beta|^{n-n_1} + 1)|\beta|^{n_1} < 2$, which is a contradiction. As for Λ_2, note that $\Lambda_2 = 0$ implies $\alpha^{2n} \in \mathbb{Q}$, which is not possible. We then get that

$$\begin{aligned}\log|\Lambda_i| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log n)\\ \times (2.6 \times 10^{12}(1 + \log n))2(\log 2)\log\alpha,\end{aligned}$$

for $i = 1, 2$. Thus,

$$\log|\Lambda_i| > -1.7 \times 10^{24}(1 + \log n)^2 \quad \text{for } i = 1, 2.$$

Comparing these with (6) and (7), we get that $(m-m_1-1)\log 2 < 1.7\times 10^{24}(1+\log n)^2$, $(n-n_1-6)\log\alpha < 1.7\times 10^{24}(1+\log n)^2$, according to whether we are in Case 1 or in Case 2. Thus, in both Case 1 and Case 2, we have

$$\begin{aligned}&\min\{(n-n_1)\log\alpha,\,(m-m_1)\log 2\} < 1.2\times 10^{12}(1+\log n),\\ &\max\{(n-n_1)\log\alpha,\,(m-m_1)\log 2\} < 1.8\times 10^{24}(1+\log n)^2.\end{aligned}\tag{11}$$

We now finally rewrite equation (2) as

$$\left|\frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1}-2^{m_1}(2^{m-m_1}-1)\right| = \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}\right| < |\beta|^{n_1} = \frac{1}{\alpha^{n_1}}.$$

Dividing both sides by $2^m-2^{m_1}$, we get

$$\begin{aligned}\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)}\right)\alpha^{n_1}2^{-m_1}-1\right| &< \frac{1}{\alpha^{n_1}(2^m-2^{m_1})} \le \frac{2}{\alpha^{n_1}2^m}\\ &\le 2\alpha^{4-n-n_1} \le \alpha^{4-n},\end{aligned}\tag{12}$$

because $\alpha^{n_1} \ge \alpha^2 > 2$. To find a lower-bound on the left-hand side, we use again Theorem 2 with the data

$$\begin{aligned}&l=3,\ \eta_1=\frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)},\ \eta_2=\alpha,\ \eta_3=2,\ d_1=1,\ d_2=n_1,\\ &d_3=-m_1,\ D=n.\end{aligned}$$

We have $\mathbb{L}=\mathbb{Q}(\sqrt{5})$ with $d_{\mathbb{L}}=2$. Using $h(x/y)\le h(x)+h(y)$ for any two nonzero algebraic numbers x and y, we have

$$\begin{aligned}h(\eta_1) &\le h\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)+h(2^{m-m_1}+1) < \log(2\sqrt{5}\alpha^{n-n_1})\\ &\quad+\log(2^{m-m_1}+1)\\ &\le (n-n_1)\log\alpha+(m-m_1)\log 2+\log(2\sqrt{5})+1 < 2\\ &\quad\times 10^{24}(1+\log n)^2,\end{aligned}$$

where in the above chain of inequalities, we used the arguments from (8) and (9) as well as the bound (11). So, we can take $A_1 = 4\times 10^{24}(1+\log n)^2$ and certainly $A_2=\log\alpha$ and $A_3 = 2\log 2$. We need to show that if we put

$$\Lambda_3 = \frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1}-2^{m_1}(2^{m-m_1}-1),$$

then $\Lambda_3\neq 0$. But $\Lambda_3=0$ leads to

$$(\alpha^{n-n_1}-1)\alpha^{n_1} = \sqrt{5}(2^m-2^{m_1}),$$

which upon conjugation in $\mathbb{L}$ leads to (10), which is impossible. Thus, $\Lambda_3\neq 0$. Theorem 2 gives

$$\begin{aligned}\log|\Lambda_3| &> -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log n)\\ &\quad\times(4\times 10^{24}(1+\log n)^2)2(\log 2)\log\alpha,\end{aligned}$$

which together with (12) gives

$$(n-3)\log\alpha < 3\times 10^{36}(1+\log n)^3,$$

leading to $n < 7\times 10^{42}$.

Now we need to reduce the bound. To do so, we make use of the following result several times, which is a slight variation of a result due to Dujella and Pethő [2] which itself is a generalization of a result of Baker and Davenport [1]. For a real number x, we put $||x|| = \min\{|x-n| : n\in\mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 3. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \le M \qquad \text{and} \qquad k \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

We return first to (5) and put

$$\Gamma = n\log\alpha - m\log 2 - \log\sqrt{5}.$$

Assume that $\min\{n-n_1, m-m_1\} \ge 20$ and we return to (5). This is not a very restrictive assumption since, as we shall see immediately, if this condition fails then we do the following:

(i) if $n-n_1 < 20$ but $m-m_1 \ge 20$, we go to (6);
(ii) if $n-n_1 \ge 20$ but $m-m_1 < 20$, we go to (7);
(iii) if both $n-n_1 < 20$ and $m-m_1 < 20$, we go to (12).

In (5), since $|e^\Gamma - 1| = |\Lambda| < 1/4$, we get that $|\Gamma| < 1/2$. Since $|x| < 2|e^x - 1|$ holds for all $x \in (-1/2, 1/2)$, we get that

$$|\Gamma| < 2\max\{\alpha^{n_1-n+6}, 2^{m-m_1+1}\} \le \max\{\alpha^{n_1-n+8}, 2^{m_1-m+2}\}.$$

Assume $\Gamma > 0$. Then

$$0 < n\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{\log(1/\sqrt{5})}{\log 2} < \max\left\{\frac{\alpha^8}{(\log 2)\alpha^{n-n_1}}, \frac{4}{(\log 2)2^{m-m_1}}\right\}.$$

We apply Lemma 3 with

$$\tau = \frac{\log\alpha}{\log 2}, \quad \mu = \frac{\log(1/\sqrt{5})}{\log 2}, \quad (A,B) = (68,\alpha) \quad \text{or} \quad (6,2).$$

We let $\tau = [a_0, a_1, a_2, \ldots] = [0, 1, 2, 3, 1, 2, 3, 2, 4, \ldots]$ be the continued fraction of τ. We take $M = 7\times 10^{42}$. We take

$$\frac{p}{q} = \frac{p_{149}}{q_{149}} = \frac{755830092745232999099612135303693391839418748444717618738467007831418529 20}{1088712850528619465432515952603697382184620103833234826296110844071070900 03}$$

where $q > 10^{74} > 6M$. We have $\varepsilon > 0.09$, therefore either

$$n - n_1 \le \frac{\log(68q/0.09)}{\log \alpha} < 369 \quad \text{or} \quad m - m_1 \frac{\log(6q/0.09)}{\log 2} < 253.$$

Thus we have that either $n - n_1 \le 368$ or $m - m_1 \le 252$. A similar conclusion is obtained when $\Gamma < 0$.

In case $n - n_1 \le 368$, we go to (6). There, we assume that $m - m_1 \ge 20$. We put

$$\Gamma_1 = n_1 \log \alpha - m \log 2 + \log\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}}\right).$$

Then (6) implies that

$$|\Gamma_1| < \frac{4}{2^{m-m_1}}.$$

Assume $\Gamma_1 > 0$. Then

$$0 < n_1 \left(\frac{\log \alpha}{\log 2}\right) - m + \frac{\log((\alpha^{n-n_1} - 1)/\sqrt{5})}{\log 2} < \frac{4}{(\log 2) 2^{m-m_1}} < \frac{6}{2^{m-m_1}}.$$

We keep the same $\tau,\ M,\ q,\ (A, B) = (6, 2)$ and put

$$\mu_k = \frac{\log((\alpha^k - 1)/\sqrt{5})}{\log 2}, \qquad k = 1, 2, \ldots, 368.$$

We have problems at $k \in \{4, 12\}$. We discard these values now and will treat them later. For the remaining values of k, we get $\varepsilon > 0.001$. Hence, by Lemma 3, we get

$$m - m_1 < \frac{\log(6q/0.001)}{\log 2} < 259.$$

Thus, $n - n_1 \le 368$ implies $m - m_1 \le 258$, unless $n - n_1 \in \{4, 12\}$. A similar conclusion is reached if $\Gamma_1 < 0$ with the same two exceptions for $n - n_1 \in \{4, 12\}$. The reason we have a problem at $k \in \{4, 8\}$ is because

$$\frac{\alpha^4 - 1}{\sqrt{5}} = \alpha^2 \quad \text{and} \quad \frac{\alpha^{12} - 1}{\sqrt{5}} = 2^3 \alpha^6.$$

So, $\Gamma_1 = (n_1 + 2)\tau - m$, or $(n_1 + 6)\tau - (m - 3)$ when $k = 4, 12$, respectively. Thus we get that

$$\left|\tau - \frac{m}{n_1 + 2}\right| < \frac{4}{2^{m-m_1}(n_1 + 2)} \quad \text{or} \quad \left|\tau - \frac{m - 3}{n_1 + 6}\right| < \frac{4}{2^{m-m_1}(n_1 + 6)}.$$

Assume $m - m_1 > 150$. Then $2^{m-m_1} > 8 \times (8 \times 10^{42}) > 8 \times (n_1 + 6)$, therefore

$$\frac{4}{2^{m-m_1}(n_1 + 2)} < \frac{1}{2(n_1 + 2)^2} \quad \text{and} \quad \frac{4}{2^{m-m_1}(n_1 + 6)} < \frac{1}{2(n_1 + 6)^2}.$$

By a criterion of Legendre, it follows that $m/(n_1+2)$ or $(m+3)/(n_1+6)$ are convergents of τ, respectively. So, say one of $m/(n_1+2)$ or $m/(n_1+6)$ is of the form p_k/q_k for some $k = 0, 1, 2, \ldots, 99$. Here we use that $q_{99} > 8 \times 10^{42} > n_1 + 6$. Then

$$\frac{1}{(a_k+2)q_k^2} < \left|\tau - \frac{p_k}{q_k}\right|.$$

Since $\max\{a_k : k = 0, \ldots, 99\} = 134$, we get that

$$\frac{1}{136q_k^2} < \frac{4}{2^{m-m_1}q_k} \quad \text{and } q_k \text{ divides one of } \{n_1+2,\ n_1+6\}.$$

Thus

$$2^{m-m_1} \le 4 \times 136(n_1+6) < 4 \times 136 \times 8 \times 10^{42}$$

giving $m - m_1 \le 151$. Hence, even in the case $n - n_1 \in \{4, 12\}$, we still keep the conclusion that $m - m_1 \le 258$.

Now let us assume that $m - m_1 \le 252$. Then we go to (7). We write

$$\Gamma_2 = n\log\alpha - m_1\log 2 + \log(1/(\sqrt{5}(2^{m-m_1}-1))).$$

We assume that $n - n_1 \ge 20$. Then

$$|\Gamma_2| < \frac{2\alpha^6}{\alpha^{n-n_1}}.$$

Assuming $\Gamma_2 > 0$, we get

$$0 < n\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log(1/(\sqrt{5}(2^{m-m_1}-1)))}{\log 2} < \frac{2\alpha^6}{(\log 2)\alpha^{n-n_1}} < \frac{52}{\alpha^{n-n_1}}.$$

We apply again Lemma 3 with the same $\tau,\ q,\ M, (A, B) = (52, \alpha)$ and

$$\mu_k = \frac{\log(1/(\sqrt{5}(2^k-1))}{\log 2} \quad \text{for } k = 1, 2, \ldots, 252.$$

We get $\varepsilon > 0.0005$, therefore

$$n - n_1 < \frac{\log(52q/0.0005)}{\log\alpha} < 379.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first get that either $n - n_1 \le 368$ or $m - m_1 \le 252$. If $n - n_1 \le 368$, then $m - m_1 \le 258$, and if $m - m_1 \le 252$, then $n - n_1 \le 378$. In conclusion, we always have $n - n_1 < 380$ and $m - m_1 < 260$.

Finally we go to (12). We put

$$\Gamma_3 = n_1\log\alpha - m_1\log 2 + \log\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)}\right).$$

Since $n \ge 400$, (12) implies that

$$|\Gamma| < \frac{2}{\alpha^{n-3}} = \frac{2\alpha^3}{\alpha^n}.$$

Assume that $\Gamma_3 > 0$. Then

$$0 < n_1\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log((\alpha^k-1)/\sqrt{5}(2^\ell-1))}{\log 2} < \frac{2\alpha^3}{(\log 2)\alpha^n} < \frac{13}{\alpha^n},$$

where $(k,l) := (n-n_1, m-m_1)$. We apply again Lemma 3 with the same $\tau,\ M,\ q,$ $(A,B) = (13,\alpha)$ and

$$\mu_{k,l} = \frac{\log((\alpha^k-1)/\sqrt{5}(2^\ell-1))}{\log 2} \quad \text{for } 1 \le k \le 379,\ 1 \le l \le 259.$$

We have a problem at $(k,l) = (4,1),\ (12,1)$ (as for the case of (6)) and additionally for $(k,l) = (8,2)$ since

$$\frac{\alpha^8-1}{\sqrt{5}(2^2-1)} = \alpha^4.$$

We discard the cases $(k,l) = (4,1),\ (12,1), (8,2)$ for the time being. For the remaining ones, we get $\varepsilon > 7\times 10^{-6}$, and hence

$$n \le \frac{\log(13q/(7\times 10^{-6}))}{\log\alpha} < 385.$$

A similar conclusion is reached when $\Gamma_3 < 0$. Hence $n < 400$. Now we look at the cases $(k,l) = (4,1),\ (12,1),\ (8,2)$. The cases $(k,l) = (4,1),\ (12,1)$ can be treated as before when we showed that $n-n_1 \le 368$ implies $m-m_1 \le 258$. The case when $(k,l) = (8,2)$ can be dealt with similarly as well. Namely, it gives

$$|(n_1+4)\tau - m_1| < \frac{13}{\alpha^n}.$$

Hence,

$$\left|\tau - \frac{m_1}{n_1+4}\right| < \frac{13}{(n_1+4)\alpha^n}. \tag{13}$$

Since $n \ge 400$, then $\alpha^n > 2\times 13\times(8\times 10^{42}) > 2\times 13(n_1+4)$, which shows that the right–hand side of inequality (13) is at most $2/(n_1+4)^2$. By Legendre's criterion, $m/(n_1+4) = p_k/q_k$ for some $k = 0, 1, \ldots, 99$. We then get by an argument similar to a previous one that

$$\alpha^n \le 13\times 136\times(8\times 10^{42})$$

which gives $n \le 220$. So we conclude that $n < 400$ holds also in the case of the pair $(k,l) = (8,2)$. However, this contradicts our working assumption that $n \ge 400$.

Theorem 1 is therefore proved. □

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References

[1] Baker A and Davenport H, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J Math. Ser. (2)* **20** (1969) 129–137

[2] Dujella A and Pethő A, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)* **49** (1998) 291–306

[3] Herschfeld A, The equation $2^x - 3^y = d$, *Bull. Amer. Math. Soc.* **41** (1935) 631

[4] Herschfeld A, The equation $2^x - 3^y = d$, *Bull. Amer. Math. Soc.* **42** (1936) 231–234

[5] Matveev E M, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000) 125–180; translation in *Izv. Math.* **64** (2000) 1217–1269

[6] Pillai S S, On $a^x - b^y = c$, *J. Indian Math. Soc. (N.S.)* **2** (1936) 119–122

[7] Pillai S S, A correction to the paper, "On $a^x - b^y = c$", *J. Indian Math. Soc.* **2** (1937) 215

[8] Stroeker R J and Tijdeman R, Diophantine equations, in: Computational methods in number theory, Part II, vol. 155 of Math. Centre Tracts, Math. Centrum (1982) (Amsterdam) pp. 321–369

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