Elliptic ovoids and their rosettes in a classical generalized quadrangle of even order

ILARIA CARDINALI^{1,*} and N S NARASIMHA SASTRY²

¹Department of Information Engineering and Mathematics, University of Siena,
Via Roma 56, I-53100, Siena, Italy

²Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,
8th Mile Mysore Road, Bangalore 560 059, India

*Corresponding author.

E-mail: ilaria.cardinali@unisi.it; nsastry@isibang.ac.in

MS received 22 October 2014; revised 19 October 2015

Abstract. Let Q_0 be the classical generalized quadrangle of order $q = 2^n$ ($n \geq 2$) arising from a non-degenerate quadratic form in a 5-dimensional vector space defined over a finite field of order q . We consider the rank two geometry $\mathcal{X}$ having as points all the elliptic ovoids of Q_0 and as lines the maximal pencils of elliptic ovoids of Q_0 pairwise tangent at the same point. We first prove that $\mathcal{X}$ is isomorphic to a 2-fold quotient of the affine generalized quadrangle $Q \setminus Q_0$, where Q is the classical (q, q^2) -generalized quadrangle admitting Q_0 as a hyperplane. Further, we classify the cliques in the collinearity graph Γ of $\mathcal{X}$. We prove that any maximal clique in Γ is either a line of $\mathcal{X}$ or it consists of 6 or 4 points of $\mathcal{X}$ not contained in any line of $\mathcal{X}$, accordingly as n is odd or even. We count the number of cliques of each type and show that those cliques which are not contained in lines of $\mathcal{X}$ arise as subgeometries of Q defined over $\mathbb{F}_2$.

Keywords. Generalized quadrangles; collinearity graph; t -fold covering; cliques.

2010 Mathematics Subject Classification. 51E12, 05B25, 05C69.

1. Introduction

We refer to [10] for the terminology and basics on finite generalized quadrangles. We only recall that a finite generalized quadrangle Q is *classical* if its points and lines are points and lines of a projective space $\text{PG}(d, q)$, the points of Q span $\text{PG}(d, q)$ and if a projective line l is a line of Q , then all points of l belong to Q . If this is the case, then the points and the lines of Q are the points and the lines of $\text{PG}(d, q)$ that are singular and totally singular, respectively, for a suitable non-degenerate alternating, hermitian or quadratic form of the underlying vector space $V(d+1, q)$ of $\text{PG}(d, q)$ (Chapter 4 of [10]).

The generalized quadrangle of order q arising from a non-degenerate symplectic form on $V(4, q)$ is denoted by $W(q)$. The generalized quadrangle of order q arising from a non-degenerate quadratic form in $V(5, q)$ is denoted by $Q(4, q)$. The (q, q^2) -generalized quadrangle arising from a non-degenerate quadratic form of Witt index 2 in $V(6, q)$ is denoted by $Q^-(5, q)$. The $(q, 1)$ -generalized quadrangle arising from a non-degenerate quadratic form of Witt index 2 in $V(4, q)$ is denoted by $Q^+(3, q)$. It is also called a *grid*.

For any q , the dual of $Q(4, q)$ is isomorphic to $W(q)$. Moreover, $Q(4, q) \cong W(q)$ if and only if q is even.

An *ovoid* of an (s, t) -generalized quadrangle $\mathcal{Q}$ is a set X of points of $\mathcal{Q}$ such that every line of $\mathcal{Q}$ meets X at exactly one point (see [10]). A simple count shows that $|X| = st + 1$. By definition, no two points of X are collinear. For a survey of ovoids of generalized quadrangles, we refer the reader to [12].

In this paper, we assume that $q = 2^n$ and denote by Q_0 a classical generalized quadrangle $Q(4, q)$ of order q embedded in a 4-dimensional projective space $\text{PG}(V_0)$, $V_0 = V(5, q)$. Let n_0 be the *nucleus* of Q_0 , i.e., n_0 is the unique non-singular point of $\text{PG}(V_0)$ with the property that the lines of $\text{PG}(V_0)$ through n_0 are the tangents to Q_0 in $\text{PG}(V_0)$ (see, e.g., Chapter 22.3 of [6]).

Classifying ovoids in Q_0 is a fundamental open problem. There are two known classes of ovoids of Q_0 , namely the class $\mathcal{E}$ of *elliptic ovoids* and the class $\mathcal{T}$ of *Tits–Suzuki ovoids*. Tits–Suzuki ovoids exist if and only if n is odd. An ovoid X of Q_0 is elliptic if and only if it spans a hyperplane of $\text{PG}(V_0)$ (whence $X = \langle X \rangle \cap Q_0$).

We recall that any two distinct elliptic ovoids of Q_0 intersect in either a singleton or a non-degenerate conic. If two elliptic ovoids meet at a single point p , we say that they are *tangent at p* .

Following [2], we define a *rosette* ρ_p of elliptic ovoids *based at a point p* as a set of q elliptic ovoids of Q_0 mutually tangent at p such that $\{X \setminus \{p\} : X \in \rho_p\}$ is a partition of the set of the points of Q_0 not collinear with p . The point p is called the *base point of ρ_p* . Clearly, a set of elliptic ovoids mutually tangent at a given point is a rosette if and only if it consists of exactly q ovoids. It is also clear that a rosette with base point p is a maximal pencil of elliptic ovoids of Q_0 mutually tangent at p .

The next Proposition summarizes some properties of elliptic ovoids and rosettes of Q_0 . Proof of Proposition 1.1(a) and (b) can be found in Theorem 3.2 of [1] (see also Example 1.4(d) of [5]). Parts (c) and (d) of Proposition 1.1 are proved in §3.

PROPOSITION 1.1

The following hold:

- (a) *Let X_1 and X_2 be two elliptic ovoids of Q_0 tangent at a point p . Then there exists a unique rosette ρ_p with base point p containing X_1 and X_2 .*
- (b) *Two elliptic ovoids of Q_0 in distinct rosettes based at the same point p meet in a conic containing p .*
- (c) *Let X_1 and X_2 be two elliptic ovoids of Q_0 tangent at a point. For each $x \in X_1 \setminus X_2$, there exists a unique elliptic ovoid of Q_0 through x which is tangent to both X_1 and X_2 .*
- (d) *Let X_1 and X_2 be two elliptic ovoids of Q_0 intersecting in a conic. If $x \in X_1 \setminus X_2$ then there exist exactly two distinct elliptic ovoids of Q_0 through x which are both tangent to X_1 and X_2 . If $x \in X_1 \cap X_2$ then there exist no elliptic ovoids of Q_0 through x which are both tangent to X_1 and X_2 . $\square$*

PROPOSITION 1.2

Let ρ_p be a rosette of elliptic ovoids of Q_0 based at a point p . Then there exists a unique projective plane π_p of $\text{PG}(V_0)$ containing p and tangent to each $X \in \rho_p$.

Proof. Let X_i, X_j, X_k be three distinct elliptic ovoids in ρ_p . Let $\pi_j := \langle X_i \rangle \cap \langle X_j \rangle$ and $\pi_k := \langle X_i \rangle \cap \langle X_k \rangle$ be the planes of the 3-dimensional subspace $\langle X_i \rangle$ spanned by X_i obtained as its intersection with the 3-dimensional subspaces spanned by X_j and X_k , respectively. Since $\langle X \rangle \cap Q_0 = X$, for every $X \in \mathcal{E}$, we have $\pi_j \cap X_i = p = \pi_k \cap X_i$.

If $\pi_j \neq \pi_k$, then π_j and π_k are two distinct planes of $\langle X_i \rangle$ both tangent to X_i at p . This is clearly impossible. Hence, $\pi_j = \pi_k$ and the proposition follows. $\square$

Let $\mathcal{X} := (\mathcal{E}, \mathcal{L})$ be the point-line geometry having the set $\mathcal{E}$ of elliptic ovoids of Q_0 as the point set and the set of rosettes of elliptic ovoids of Q_0 as the line set. An elliptic ovoid X is incident with a rosette ρ_p if $X \in \rho_p$. The geometry $\mathcal{X}$ is a semi-partial geometry with parameters $(s, t, \alpha, \mu) = (q-1, q^2, 2, 2q(q-1))$ (see [5] and [4]). A very straightforward proof of this claim will also be given in §2.

1.1 Main results

Let $Q \cong Q^-(5, q)$ be a classical (q, q^2) -generalized quadrangle embedded in the 5-dimensional projective space $\text{PG}(V)$, where $V = V(6, q)$. We may assume that $Q_0 = Q \cap H_0$ for a suitable hyperplane H_0 of $\text{PG}(V)$. Thus, Q_0 is embedded in $H_0 \cong \text{PG}(4, q)$. Let $\perp$ denote the orthogonality relation defined by Q in $\text{PG}(V)$. The point $n_0 := H_0^\perp$ is the nucleus of the quadric Q_0 .

Let $\hat{\mathcal{X}} := (\hat{P}, \hat{\mathcal{L}})$ be the point-line geometry having the set $\hat{P}$ of the points of $Q \setminus Q_0$ as the point set and the set $\hat{\mathcal{L}} := \{l \setminus Q_0 : l \text{ a line of } Q \text{ not contained in } Q_0\}$ as the line set. This geometry is called *affine generalized quadrangle*.

If l is a line of Q not contained in Q_0 , then $l \cap Q_0$ is a point, henceforth denoted by l^∞ . We denote by $\hat{l}$ the line of $\hat{\mathcal{X}}$ corresponding to l and we call l^∞ the *point at infinity* of $\hat{l}$. Thus,

$$\hat{l} = l \setminus \{l^\infty\} \quad \text{and} \quad l = \hat{l} \cup \{l^\infty\}. \quad (1)$$

The incidence relation in $\hat{\mathcal{X}}$ is inherited from the incidence relation of Q . It is easy to see that the incidence graph of $\hat{\mathcal{X}}$ is connected. It has girth 8, whence $\hat{\mathcal{X}}$ has gonality 4. The collinearity graph $\hat{\Gamma}$ of $\hat{\mathcal{X}}$ is regular and has diameter 3. It contains 4-circuits and every clique of $\hat{\Gamma}$ is contained in a line of $\hat{\mathcal{X}}$ (see, e.g., §8.4.1 of [9]).

We recall that a *morphism* of rank 2 geometries is a morphism of their incidence graphs (see, e.g., Chapter 8 of [9]). Given two rank 2 geometries $\mathcal{G}_1$ and $\mathcal{G}_2$, a morphism $\varphi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a *covering* from $\mathcal{G}_1$ to $\mathcal{G}_2$ if, for any point p of $\mathcal{G}_1$ and any line l of $\mathcal{G}_1$, φ induces a bijection from the point set of l to the point set of $\varphi(l)$ and from the set of lines of $\mathcal{G}_1$ incident with p to the set of lines of $\mathcal{G}_2$ incident with $\varphi(p)$. If each fiber of the covering has size t , then we say that φ is a *t-fold covering* of $\mathcal{G}_2$, $\mathcal{G}_1$ is called a *t-fold cover* of $\mathcal{G}_2$ and $\mathcal{G}_2$ is called a *t-fold quotient* of $\mathcal{G}_1$. Note that all coverings of connected geometries are surjective (see §8.3 of [9]). In the terminology of graph theory, a covering of geometries is a covering of their incidence graphs. It also induces a covering of the collinearity graphs, satisfying the additional property that, if a, b, c are different points of $\mathcal{G}_1$ with b and c collinear with a and $\varphi(a), \varphi(b), \varphi(c)$ belonging to a common line of $\mathcal{G}_2$, then a, b, c belong to the same line of $\mathcal{G}_1$. We warn that, however, if b and c are collinear with a and, $\varphi(b)$ and $\varphi(c)$ are mutually collinear but not on the same line as $\varphi(a)$, then b and c need not be collinear in $\mathcal{G}_1$.

In Section 2, we prove:

Theorem 1. *The geometry $\hat{\mathcal{X}}$ is a 2-fold cover of the geometry $\mathcal{X}$. Explicitly, let $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ map $x \in \hat{P}$ to $x^\perp \cap Q_0$ and $\hat{l} \in \hat{L}$ to ρ_p , where $p = l^\infty$. Then φ_0 is a 2-fold covering of $\mathcal{X}$.*

The covering φ_0 defined above is called the *canonical covering* from $\hat{\mathcal{X}}$ to $\mathcal{X}$. As Q can be recovered from $\hat{\mathcal{X}}$ (see Theorem 2 of [11]), every automorphism $\hat{\alpha}$ of $\hat{\mathcal{X}}$ can be extended to a unique automorphism α of Q stabilizing Q_0 . Let α_0 be the automorphism of Q_0 induced by α . Clearly, $\varphi_0 \circ \hat{\alpha} = \alpha_0 \circ \varphi_0$. It follows that the composition $\varphi := \varphi_0 \circ \hat{\alpha}$ is still a 2-fold covering of $\mathcal{X}$. Perhaps, all 2-fold coverings from $\hat{\mathcal{X}}$ to $\mathcal{X}$ arise in this way, but we will not pursue the conjecture in this paper.

In graph theory the classification and characterization of the maximal cliques of a graph is one of the most interesting and difficult problems to address. For the graph we are interested in, relying on the underlying geometry, where we succeed in both the classification and characterization. Indeed, in §3, we determine the cliques of the collinearity graph Γ of $\mathcal{X}$. A clique of Γ is *linear* if it is a subset of a line of $\mathcal{X}$; *non-linear* otherwise. A clique of size i is called an *i-clique*.

As we will prove in §3 (Proposition 3.2), a linear clique and a non-linear clique meet in at most two vertices.

Theorem 2. *Let $q = 2^n > 2$. The following hold:*

- (a) *If n is even, then a maximal clique of Γ has either q or 4 elements. Further, q -cliques are linear and maximal 4-cliques are non-linear.*
- (b) *If n is odd, then a maximal clique of Γ has either q or 6 elements. Further, q -cliques are linear and maximal 6-cliques are non-linear.*
- (c) *Every non-linear 3-clique can be extended to $q + 1$ distinct non-linear 4-cliques.*
- (d) *If n is odd, then*
 - (i) *every non-linear 4-clique extends precisely to two non-linear 5-cliques;*
 - (ii) *every non-linear 5-clique extends uniquely to a non-linear 6-clique.*

We remark that Γ is a complete graph on 6 vertices if $q = 2$. If $q = 4$, all maximal cliques have size 4. Denote by $\hat{\Gamma}$ the collinearity graph of $\hat{\mathcal{X}}$.

A *hexagon* $H = (a^1, b^1, c^1, a^2, b^2, c^2, a^1)$ of $\hat{\mathcal{X}}$ is a 6-circuit of $\hat{\Gamma}$ (figure 1). Note that H admits a bipartition by two 3-sets $\{a^1, b^1, c^1\}$ and $\{a^2, b^2, c^2\}$ as well as a partition by three 2-sets $\{a^1, a^2\}$, $\{b^1, b^2\}$ and $\{c^1, c^2\}$ such that vertices in the same 2-set are on a secant line to Q through n_0 . Vertices in the same 2-set are called *opposite*. We say that a hexagon is *centric* if all projective lines through opposite vertices are concurrent. The point of concurrence is called the *center of the hexagon*.

A *cube* C of $\hat{\mathcal{X}}$ is an induced subgraph of $\hat{\Gamma}$ with 8 vertices as in figure 2, where adjacents are represented by thick lines.

The graph C admits a partition by two 4-sets $\{a^1, b^1, c^1, d^1\}$, $\{a^2, b^2, c^2, d^2\}$ as well as a partition by four 2-sets $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$, two vertices in a 2-set being called *opposite*. We say that a cube is *centric* if all projective lines through opposite vertices are concurrent. The point of concurrence is called the *center of the cube*.

Note that a hexagon (respectively, a cube) is the complement of a (3×2) -grid graph (respectively, a (4×2) -grid graph). The bipartition in two classes of size 3 (respectively, 4)

Figure 1. Centric hexagon.

Figure 2. Centric cube.

is one of the two families of lines of the grid and the partition by three (respectively four) 2-sets is the other family. We exploit this observation to define dodecades. Consider a (6×2) -grid as follows:

A *dodecade* of $\hat{\mathcal{X}}$ is an induced subgraph of $\hat{\Gamma}$ with 12 vertices

$$D = \{a^1, b^1, c^1, d^1, e^1, f^1, a^2, b^2, c^2, d^2, e^2, f^2\}$$

isomorphic to the complement of the collinearity graph of a grid as in figure 3. Clearly, D admits a bipartition into two classes of size 6 (corresponding to the two long lines of the grid) as well as a partition into six 2-sets, corresponding to the six short lines. Two points in the same 2-set are said to be *opposite*. Clearly, any four (respectively, three) pairs of opposite vertices of a dodecade form a cube (respectively, a hexagon). We say that a dodecade is *centric* if all the projective lines through opposite vertices are concurrent. The point of concurrence is called the *center of the dodecade*.

Let $\varphi = \varphi_0 \circ \hat{\alpha}$, where $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ is the canonical 2-fold covering of $\mathcal{X}$ as in Theorem 1 and $\hat{\alpha} \in \text{Aut}(\hat{\mathcal{X}})$.

For the subgraph Γ_A of Γ induced on a subset A of the vertex set of $\mathcal{X}$, the subgraph $\hat{\Gamma}_A$ of $\hat{\Gamma}$ induced on the subset $\varphi^{-1}(A)$ of the vertex set of $\hat{\Gamma}$ is called the *preimage* of Γ_A by φ .

Given a path $(X_0, X_1, \dots, X_n)$ in Γ and a vertex $x_0 \in \varphi^{-1}(X_0)$, there exists a unique path $(x_0, x_1, \dots, x_n)$ in $\hat{\Gamma}$ such that $\varphi(x_i) = X_i$ for every $i = 0, 1, \dots, n$. In graph theory and topology, the path $(x_0, x_1, \dots, x_n)$ is called the *lifting* of $(X_0, X_1, \dots, X_n)$ at x_0 through φ . By a harmless abuse of terminology, we adapt this terminology

Figure 3. (6×2) -grid as complement of a dodecade.

to the induced subgraph Γ_A of Γ , calling *liftings* of Γ_A the connected components of $\hat{\Gamma}_A$.

The following theorem characterizes the cliques of Γ in terms of their liftings. Recall that n_0 is the nucleus of Q_0 .

Theorem 3. *Suppose $q \geq 4$. Let $\varphi = \varphi_0 \circ \hat{\alpha}$ with φ_0 and $\hat{\alpha}$ as above. The following hold:*

- (a) *The covering φ induces a bijection between the set of all centric hexagons of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 3-cliques in Γ .*
- (b) *The covering φ induces a bijection between the set of all centric cubes of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 4-cliques in Γ .*
- (c) *The covering φ induces a bijection between the set of all centric dodecades of $\hat{\Gamma}$ with center n_0 and the set of all non-linear 6-cliques in Γ .*

Let S be a subset of $\text{PG}(V)$. According to Theorem 1.1 of [3], there exists a unique subfield $\mathbb{F}_S$ of $\mathbb{F}_q$ such that S is contained in a projective subgeometry of $\text{PG}(V)$ defined over $\mathbb{F}_S$ and is minimal with respect to this property. Moreover, the family of projective subgeometries of $\text{PG}(V)$ defined over $\mathbb{F}_S$ and containing S admits a smallest member which we shall denote by $\langle S \rangle_{\mathbb{F}_S}$. We call $\langle S \rangle_{\mathbb{F}_S}$ the $\mathbb{F}_S$ -span of S .

If S is a subset of Q , we define the *subgeometry of Q induced on S* to be the pair $Q(S) := (S, \mathcal{L}_S)$, where $\mathcal{L}_S := \{l \cap S : l \text{ is a line of } Q \text{ such that } |l \cap S| \geq 2\}$.

As recalled a few lines above, there exists a unique smallest projective subgeometry $\langle S \rangle_{\mathbb{F}_S}$ of $\text{PG}(V)$ containing S and defined over a subfield $\mathbb{F}_S$ of $\mathbb{F}_q$. We say that $Q(S)$ is an $\mathbb{F}_2$ -subgeometry of Q if $\mathbb{F}_S = \mathbb{F}_2$.

The next theorem, combined with Theorem 3, gives a characterization of non-linear i -cliques, $i \in \{3, 4, 6\}$, in terms of $\mathbb{F}_2$ -subgeometries of Q .

Theorem 4. *Suppose $q \geq 4$. The following hold:*

- (a) *A centric hexagon of $\hat{\mathcal{X}}$ with center n_0 is the subgraph of $\hat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in an $\mathbb{F}_2$ -subgeometry $Q(S) \cong Q^+(3, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*
- (b) *A centric cube of $\hat{\mathcal{X}}$ with center n_0 is the subgraph of $\hat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in an $\mathbb{F}_2$ -subgeometry $Q(S) \cong Q(4, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*
- (c) *A centric dodecade of $\hat{\mathcal{X}}$ with center n_0 is the subgraph of $\hat{\Gamma}$ induced on the set-complement $Q(S) \setminus H_0$ of $H_0 = n_0^\perp$ in a $\mathbb{F}_2$ -subgeometry $Q(S) \cong Q^-(5, 2)$ of Q such that $n_0 \in \langle S \rangle_{\mathbb{F}_2}$.*

The relevance of the parity of n for the existence of maximal non-linear 6-cliques in Theorem 2 is justified by Theorems 3 and 4. Indeed, every non-linear 6-clique of Γ arises as the complement of H_0 in a $(2, 4)$ -subquadrangle $Q^-(5, 2)$ of Q (Theorem 4(c)), but it is an easy exercise to show that $Q^-(5, 2)$ is a subgeometry of $Q^-(5, 2^n)$ if and only if n is odd.

Each non-linear 4-clique of Γ arises as the complement of a tangent hyperplane in the $(2, 2)$ -subquadrangle $Q(4, 2)$ of $Q^-(5, 2^n)$. Observe that $Q(4, 2)$ is a subgeometry of $Q^-(5, 2^n)$ for each n . So, non-linear 4-cliques exist for all n .

2. The semi-partial geometry $\mathcal{X}$

We keep the terminology and notation of §1.

2.1 The quotient geometry $\hat{\mathcal{X}}/R$

Let R be the equivalence relation on the point set $\hat{P}$ of the affine generalized quadrangle $\hat{\mathcal{X}}$ defined as follows: for $x, y \in \hat{P}$, define xRy if and only if either $x = y$ or $x^\perp \cap Q_0 = y^\perp \cap Q_0 (= x^\perp \cap y^\perp)$, where $\perp$ is the orthogonality relation defined by Q , as in §1.1 (see §8.4 of [9]). Note that distinct points of $\hat{\mathcal{X}}$ are R -equivalent if and only if they have distance 3 in the collinearity graph $\hat{\Gamma}$ of $\hat{\mathcal{X}}$.

For $x \in \hat{P}$ and $\hat{l} \in \hat{L}$, let $[x] = \{y \in \hat{P} : xRy\}$ and $[\hat{l}] := \{[x] : x \in \hat{l}\}$. Note that $|[x]| = 2$ and $|[\hat{l}]| = q$. Given any two distinct lines $\hat{l}$ and $\hat{m}$ of $\hat{\mathcal{X}}$, we put $\hat{l}R\hat{m}$ if and only if $[\hat{l}] = [\hat{m}]$.

We define the *quotient geometry* $\hat{\mathcal{X}}/R$ as the point-line geometry having $\{[x] : x \in \hat{P}\}$ as the point set and $\{[\hat{l}] : \hat{l} \in \hat{L}\}$ as the line set. A point $[x]$ is incident with a line $[\hat{l}]$ if $x \in \hat{l}$.

We now give an equivalent description of the quotient geometry $\hat{\mathcal{X}}/R$. We recall some well-known facts about the action of the polarity defined by $\perp$ on the subspaces of $\text{PG}(V)$ (see, e.g., Chapter 22.3 of [6]).

- If x is a point of Q , then $x^\perp$ is a hyperplane tangent to Q at x .
- If S is a 3-dimensional subspace of $\text{PG}(V)$ such that $S \cap Q$ is an elliptic quadric in S , then $S^\perp$ is a line secant to Q .
- If S is a 3-dimensional subspace of $\text{PG}(V)$ such that $S \cap Q$ is a hyperbolic quadric in S , then $S^\perp$ is a line disjoint from Q .
- If π_p is a projective plane of $\text{PG}(V)$ such that $\pi_p \cap Q = \{p\}$, then $\pi_p^\perp$ is a projective plane intersecting Q in two lines meeting at p . Moreover, $\pi_p \cap \pi_p^\perp = \{p\}$ and $\langle \pi_p, \pi_p^\perp \rangle = p^\perp$.

Let ν be the unique non-trivial elation of $\text{PG}(V)$ stabilizing Q and having the nucleus n_0 of Q_0 as the center and the hyperplane $H_0 = n_0^\perp$ as the axis. Clearly, ν is an involution. The orbits of ν on the point set of $\hat{\mathcal{X}}$ are precisely the equivalence classes of R on $\hat{P}$. Hence, for any point x of $\hat{P}$, the R -equivalence class $[x]$ is $\{x, \nu(x)\}$ obtained as $\langle x, n_0 \rangle \cap Q$, where $\langle x, n_0 \rangle$ is the line (secant to Q) of $\text{PG}(V)$ through x and n_0 . Accordingly, a line $[\hat{l}]$ of $\hat{\mathcal{X}}/R$ corresponds to the set $(l \cup \nu(l)) \setminus (l \cap \nu(l))$ obtained as $\langle l, \nu(l) \rangle \cap Q$ minus the point $l \cap \nu(l)$, where $l = \hat{l} \cup \{l^\infty\}$ (see (1)). Note also that $l \cap \nu(l) = l^\infty$ and $\langle l, \nu(l) \rangle = \langle l, n_0 \rangle$.

2.2 Proof of Theorem 1

According to the definition of t -fold covering given in §1.1 and the facts recalled in §2.1, it is easy to see that the projection $\pi : \hat{\mathcal{X}} \rightarrow \hat{\mathcal{X}}/R$ mapping any point x of $\hat{\mathcal{X}}$ to the point $[x]$ of $\hat{\mathcal{X}}/R$ and any line l of $\hat{\mathcal{X}}$ to the line $[l]$ of $\hat{\mathcal{X}}/R$ is a 2-fold covering. Hence

Lemma 2.1. $\hat{\mathcal{X}}$ is a 2-fold cover of $\hat{\mathcal{X}}/R$. □

Lemma 2.2. The geometries $\hat{\mathcal{X}}/R$ and $\mathcal{X}$ are isomorphic.

Proof. Let $\psi : \mathcal{X} \rightarrow \hat{\mathcal{X}}/R$ be the map defined on the points and the lines of $\mathcal{X}$ as follows: for $X \in \mathcal{E}$, $X^\perp$ is a line of $\text{PG}(5, q)$ through n_0 , secant to Q . We set $\psi(X) = X^\perp \cap Q$. For a line ρ_p of $\mathcal{X}$, let π_p be the unique tangent plane to all $X \in \rho_p$ (see Proposition 1.2). Then, $\pi_p^\perp$ is a plane of $\text{PG}(5, q)$ intersecting Q in two lines l, l' such that $l \cap Q_0 = l' \cap Q_0 = \{p\}$. Then, $\psi(\rho_p) = [\hat{l}] = [\hat{l}']$. It is easily seen that ψ is an isomorphism of geometries. □

Theorem 1 follows from Lemmas 2.1 and 2.2, in particular, $\varphi_0 := \psi^{-1} \circ \pi$ where φ_0 is the canonical covering of $\mathcal{X}$ as defined in Theorem 1.

3. The collinearity graph of $\hat{\mathcal{X}}$

In this section we will investigate properties of the collinearity graph Γ of $\mathcal{X}$. Recall that the vertices of Γ are the elliptic ovoids of Q_0 and two vertices X_1 and X_2 are adjacent if and only if they are tangent, i.e. $|X_1 \cap X_2| = 1$.

If $q = 2$, then Γ is a complete graph on 6 vertices, $\hat{\Gamma}$ is the set complement of a (2×6) -grid and the R -equivalence classes are the short lines of the grid.

From now on, we assume that $q \geq 4$.

PROPOSITION 3.1

If $q > 2$, then the collinearity graph Γ of $\mathcal{X}$ has diameter 2.

Proof. The collinearity graph $\hat{\Gamma}$ of $\hat{\mathcal{X}}$ has diameter 3 and the R -equivalence classes are the pairs of points of $\hat{\mathcal{X}}$ at mutual distance 3 (see §8.4 of [9]). Hence $\hat{\Gamma}/R$ has diameter at most 2. In fact, it is easy to see that it has diameter exactly 2 except when $q = 2$. By Lemma 2.2, the graph Γ has diameter 2. □

In [7], it is proved that Γ is a strongly regular graph with parameters $v = q^2(q^2 - 1)/2$, $k = (q - 1)(q^2 + 1)$, $\lambda = (q - 1)(q + 2)$ and $\mu = 2q(q - 1)$.

Relying on properties of the canonical covering φ_0 , we can immediately deduce the parameters (v, k, λ, μ) of Γ and hence, in light of Proposition 3.1, reprove that Γ is a strongly regular graph. Indeed,

v : The number of vertices of Γ is $|\hat{P}|/2$.

k : The number of vertices adjacent to a given vertex X of Γ is $|\{\text{lines of } \hat{\mathcal{X}} \text{ through } x\}| \times |\{\text{points different from } x \text{ on any line of } \hat{\mathcal{X}} \text{ through } x\}|$, for a point $x \in X^\perp \cap \hat{P}$.

λ : Let X and Y be two adjacent vertices of Γ , i.e., X and Y are two elliptic ovoids of Q_0 tangent at some point $p \in Q_0$. They define a line ρ_p of $\mathcal{X}$. Then, $\varphi_0^{-1}(\rho_p) = \pi_p^\perp \cap Q = \{l^1, l^2\} \setminus \{p\}$, where l^1, l^2 are lines of Q through p , $\varphi_0^{-1}(X) = \{x^1, x^2\} \subset \varphi_0^{-1}(\rho_p)$ and $\varphi_0^{-1}(Y) = \{y^1, y^2\} \subset \varphi_0^{-1}(\rho_p)$. Suppose that x^1 and y^1 are on $\hat{l}^1$ and that x^2 and y^2 are on $\hat{l}^2$. The number of vertices of Γ adjacent to both X and Y is the same as the number $q - 2$ of points $z \in \hat{P}$ on $\hat{l}^1 \setminus \{x^1, y^1\}$ (equivalently, the number of points on $\hat{l}^2 \setminus \{x^2, y^2\}$).

These points bijectively correspond to the ovoids of ρ_p different from X and Y . We also have to count the number of vertices of Γ adjacent to both X and Y but not on the same line ρ_p of $\mathcal{X}$ defined by X and Y . This number is equal to the number of points $z \in \hat{\mathcal{X}}$ at distance 1 from both x^1 and y^2 (equivalently, at distance 1 from both x^2 and y^1). There are q^2 such points. Indeed, on any line of $\hat{\mathcal{X}}$ through x^1 different from $\hat{l}^1$, there is a unique point of $\hat{\mathcal{X}}$ collinear to y^2 . Hence, $\lambda = q^2 + q - 2$.

μ : Let X and Y be two non-adjacent vertices of Γ , i.e., X and Y are two elliptic ovoids of Q_0 intersecting in a conic. Suppose $\varphi_0^{-1}(X) = \{x^1, x^2\}$ and $\varphi_0^{-1}(Y) = \{y^1, y^2\}$. Note that there are no elliptic ovoids Z tangent to both X and Y at a point in $X \cap Y$ because, otherwise, X and Y would be on the same line of $\mathcal{X}$, which is not possible because X and Y are not adjacent in Γ . Thus, the number of vertices of Γ adjacent to both X and Y is the same as the number of points $z \in Q \setminus Q_0$ not collinear with any point of $X \cap Y$, but collinear to x^1 and y^2 or to x^2 and y^1 . Let $\hat{l}$ be a line of $\hat{\mathcal{X}}$ through y^1 with $l^\infty \in Y \setminus X$. There are $q^2 - q$ such lines. For each of these $q^2 - q$ lines, there are exactly two distinct points adjacent to only one of x^1 or x^2 . Hence $\mu = 2(q^2 - q)$.

In the above calculation of the parameters λ and μ , we have also proved two interesting properties of the geometry $\mathcal{X}$ which are stated as part (c) and (d) of Proposition 1.1.

3.1 Cliques of Γ and their liftings

In this section we classify cliques in Γ . According to §1.1, a clique in Γ consisting of members of $\mathcal{X}$ sharing the same point p is a linear clique (based at p). A rosette of $\mathcal{X}$ based at p , being a maximal pencil of elliptic ovoids mutually tangent at p , is a maximal linear clique (this is part of Claims (a) and (b) of Theorem 2).

Let $\hat{\Gamma}$ denote the collinearity graph of the affine quadrangle $\hat{\mathcal{X}}$, as in §1. Let $\varphi_0: \hat{\mathcal{X}} \rightarrow \mathcal{X}$ be the canonical covering as in Theorem 1. The fibers of φ_0 are the pairs of points of $\hat{\mathcal{X}}$ at mutual distance 3 (see §2.1). We will adopt the following notation. If X is a vertex of Γ , then $\varphi^{-1}(X) = \{x^1, x^2\} (= X^\perp \cap Q)$.

3.2 Preimages of edges and linear cliques

The preimage of an edge in Γ is the union of two disjoint edges in $\hat{\Gamma}$ which are the liftings of that edge. Given a line ρ_p of $\mathcal{X}$, we have $\varphi_0^{-1}(\rho_p) = \hat{l} \cup \hat{m}$ for two lines $\hat{l}, \hat{m}$ of $\hat{\mathcal{X}}$ such that $l^\infty = m^\infty = p$. The preimage of the subgraph Γ_{ρ_p} induced by Γ on ρ_p is the subgraph induced by $\hat{\Gamma}$ on $\hat{l} \cup \hat{m}$. This graph is the disjoint union of two complete subgraphs, namely $\hat{l}$ and $\hat{m}$.

PROPOSITION 3.2

Let $\mathcal{C}$ be an i -clique of Γ , $i \geq 4$, containing a linear 3-clique. Then $\mathcal{C}$ is a linear clique.

Proof. Let A, B, C, X be four distinct vertices of $\mathcal{C}$, with A, B, C forming a linear subclique and let ρ be the line of $\mathcal{X}$ on $\{A, B, C\}$.

By way of contradiction, suppose that $X \notin \rho$. Let us lift the ordered closed configuration (X, A, B, C, X) of Γ starting at x^1 recalling that A, B, C are on the same line ρ of $\mathcal{X}$. Since φ_0 is a covering, the point x^1 is adjacent in $\hat{\Gamma}$ with exactly one preimage of A , say a^1 . Similarly, a^1 is adjacent with exactly one preimage b^1 of B and b^1 is adjacent

with exactly one preimage c^1 of C . Again, c^1 is adjacent with exactly one preimage x^2 of X . Moreover, since φ_0 is a covering, $\varphi_0^{-1}(\rho)$ contains exactly one line $\hat{l}$ of $\hat{\mathcal{X}}$ incident with all of a^1, b^1, c^1 . On the other hand, $x^1, x^2 \notin \hat{l}$ as $X \notin \rho$.

Since there are no triangles in $\hat{\mathcal{X}}$, $x^1 \neq x^2$. The path $(x^1, a^1, b^1, c^1, x^2)$ is the lifting of (X, A, B, C, X) at x^1 .

Lift now the closed ordered path (X, A, B, X) of Γ starting at x^1 . Proceeding in the same way as above, we get a path (x^1, a^1, b^1, x^3) where the vertex x^3 must be different from x^1 because there are no triangles in $\hat{\mathcal{X}}$. Similarly, (x^2, c^1, b^1, x^3) is the lifting of (X, C, B, X) at x^2 . Hence $x^3 \neq x^2$ (see figure 4).

It follows that $\varphi_0^{-1}(X)$ contains at least three distinct points, namely x^1, x^2, x^3 . This is a contradiction because φ_0 is a 2-fold covering. $\square$

3.3 Liftings of non-linear cliques

We establish the following notation for projective points: $a = [v] = [v_1, \dots, v_n]$, where $v = (v_1, \dots, v_n)$ is a given vector representing the projective point a .

Given two projective points a and b , symbols such as $ta + sb$ with t and s scalars are nonsense. However, when particular representatives v and w have been chosen for a and b , we will take the liberty of writing $ta + sb$ for $[tv + sw]$. Similarly, given a quadratic form f , we write $f(a)$ for $f(v)$. Of course, these are notational abuses but they are harmless as far as it is clear which is the representative vector chosen for a given point. In particular, when doing so, we forbid ourselves any rescaling of the given vectors.

Suppose that the (q, q^2) -generalized quadrangle Q is represented, with respect to a given basis, by the equation $f(v) = 0$, where the coordinates of v with respect to the given basis are $(x_i)_{i=1}^6$ and

$$f((x_1, x_2, x_3, x_4, x_5, x_6)) := x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + \lambda x_6^2 \quad (2)$$

with λ a given element of $\mathbb{F}_q$ with $\text{Tr}(\lambda) = 1$. Recall that the trace function $\text{Tr}: \mathbb{F}_q \rightarrow \mathbb{F}_2$, $\text{Tr}(x) = \sum_{i=0}^{n-1} x^{2^i}$, ($q = 2^n$) is an $\mathbb{F}_2$ -linear surjective homomorphism and a polynomial $x^2 + xy + \lambda y^2$ is irreducible over $\mathbb{F}_q$ if and only if $\text{Tr}(\lambda) = 1$.

The nucleus n_0 of Q_0 is represented by the vector $(0, 0, 0, 0, 0, 1)$. Accordingly, $H = n_0^\perp$ is the hyperplane defined by the equation $x_6 = 0$.

We first prove a technical lemma.

Lemma 3.3. *Let λ, μ be two elements of $\mathbb{F}_q$. Suppose that $\text{Tr}(\lambda) = 1$ and $\mu \neq 1$. Then the equation $x^2 + xy + \lambda y^2 + \mu = 1$, in the unknowns x, y , admits $q + 1$ distinct solutions in $\mathbb{F}_q^2$.*

Figure 4. Argument in the proof of Proposition 3.2.

Proof. If $y = 0$, then the pair $(x, y) = (\mu^{2^{n-1}} + 1, 0)$ is the unique solution of $x^2 + xy + \lambda y^2 + \mu = 1$. If $y \neq 0$, then the equation $x^2 + xy + \lambda y^2 + \mu = 1$ is soluble if and only if the equation $t^2 + t + \lambda + (\mu + 1)/y^2 = 0$ is soluble in the unknown $t := x/y$. This latter requirement is equivalent to $\text{Tr}(\lambda + (\mu + 1)/y^2) = 0$. Since

$$\text{Tr}(\lambda + (\mu + 1)/y^2) = \text{Tr}(\lambda) + \text{Tr}((\mu + 1)/y^2) = 1 + \text{Tr}((\mu + 1)/y^2),$$

the equation $t^2 + t + (\lambda + (\mu + 1)/y^2) = 0$ admits solutions if and only if $\text{Tr}((\mu + 1)/y^2) = 1$.

As the trace function Tr is a surjective homomorphism, there are $q/2$ distinct values of y such that $\text{Tr}((\mu + 1)/y^2) = 1$ (see e.g. [8]). For each of these $q/2$ values of y , there are two distinct solutions of $t^2 + t + \lambda + (\mu + 1)/y^2 = 0$. So, if $y \neq 0$, there are $2 \cdot \frac{q}{2}$ distinct pairs (x, y) which satisfy $x^2 + xy + \lambda y^2 + \mu = 1$. $\square$

3.3.1 Non-linear i -cliques, $i \in \{3, 4\}$. Hexagons, cubes and dodecades of $\hat{\mathcal{X}}$ and their centers, if any, have been defined in §1.1. Clearly, those definitions can be immediately generalized to Q , by replacing $\hat{\Gamma}$ with the collinearity graph of Q . In particular, a hexagon (cube, dodecade) of Q is *centric* if the projective lines joining pairs of opposite points are concurrent in one point, called the *center* of the hexagon (cube, dodecade). These more general notions will be used from time to time in the sequel.

Theorem 3.4. *The preimage in $\hat{\Gamma}$ of a non-linear 3-clique of Γ under φ_0 is a centric hexagon of $\hat{\mathcal{X}}$ with the nucleus n_0 of Q_0 as the center.*

Proof. Let $\mathcal{C} = \{A, B, C\}$ be a non-linear 3-clique of Γ . The lifting of the cycle (A, B, C, A) of $\mathcal{C}$ starting at $a^1 \in \varphi_0^{-1}(A)$ is the path (a^1, b^1, c^1, a^2) of $\hat{\Gamma}$. Since $\hat{\mathcal{X}}$ has no proper triangles, $a^1 \neq a^2$. The lifting of (A, B, C, A) at a^2 is the path (a^2, b^2, c^2, a^3) , where $b^2 \neq b^1$ and $c^2 \neq c^1$ because there are no proper triangles in $\hat{\mathcal{X}}$ and a^3 is a preimage of A , different from a^2 . As $|\varphi_0^{-1}(A)| = 2$, necessarily $a^3 = a^1$. Now, by pasting $\{a^1, b^1, c^1, a^2\}$ with $\{a^2, b^2, c^2, a^1\}$ we obtain a proper hexagon H . As φ_0 is a 2-fold covering, $H = \varphi_0^{-1}(\mathcal{C})$, that is, $\mathcal{C}$ lifts to H .

By definition of φ_0 (see Theorem 1), each of the pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$ is on a line secant to Q through n_0 . $\square$

Every centric hexagon of $\hat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 3-clique of Γ . This, combined with Theorem 3.4, proves Theorem 3(a). $\square$

Theorem 3.5. *The preimage of a non-linear 4-clique of Γ under φ_0 is a centric cube of $\hat{\mathcal{X}}$ with the nucleus n_0 of Q_0 as the center.*

Proof. Let $\mathcal{C} = \{A, B, C, D\}$ be a non-linear 4-clique of Γ . Let $\varphi_0^{-1}(A) = \{a^1, a^2\}$, $\varphi_0^{-1}(B) = \{b^1, b^2\}$, $\varphi_0^{-1}(C) = \{c^1, c^2\}$ and $\varphi_0^{-1}(D) = \{d^1, d^2\}$.

The lifting of the closed path (A, B, C, D, A) of $\mathcal{C}$ starting at a^1 is a path $\mathcal{C}^1 = (a^1, b^1, c^1, d^1, a^i)$, where $i = 1$ or $i = 2$. Suppose $i = 2$. Then, the lifting of the closed path (A, B, C, A) starting at a^1 is either (a^1, b^1, c^1, a^1) or (a^1, b^1, c^1, a^2) . None

of the latter liftings is possible, else we get a proper triangle in $\hat{\mathcal{X}}$. Hence $i = 1$, namely $\mathcal{C}^1 = (a^1, b^1, c^1, d^1, a^1)$.

Since $\hat{\mathcal{X}}$ has no proper triangles, the following hold:

- (a) the lifting of (A, B, C, A) starting at a^1 is (a^1, b^1, c^1, a^2) ;
- (b) the lifting of (A, C, D, A) starting at a^1 is (a^1, c^1, d^1, a^2) ;
- (c) the lifting of (B, C, A, B) starting at b^1 is (b^1, c^1, a^2, b^2) ;
- (d) the lifting of (B, D, A, B) starting at b^1 is (b^1, d^1, a^2, b^2) ;
- (e) the lifting of (D, B, C, D) starting at d^1 is (d^1, b^2, c^2, d^2) .

Thus, we get the cube as in figure 2.

By definition of φ_0 , each of the pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$ is on a line secant to Q through n_0 . $\square$

Every centric cube of $\hat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 4-clique of Γ . This, combined with Theorem 3.5, proves Theorem 3(b). $\square$

A set of four distinct points $\{a, b, c, d\}$ of Q is a *quadrangle* of Q if it is the set of points of a $(1, 1)$ -subquadrangle of Q , that is,

$$f(a) = f(b) = f(c) = f(d) = 0; \quad a \perp b \perp c \perp d \perp a; \quad a \not\perp c; \quad b \not\perp d. \quad (3)$$

Let $\mathcal{F}$ be a quadrangle of Q . We denote by $I(\mathcal{F})$ the set of points $p \in \text{PG}(V) \setminus Q$ such that $p^\perp \cap \mathcal{F} = \emptyset$ and there exists a cube in the affine generalized quadrangle $Q \setminus p^\perp$ with center p having $\mathcal{F}$ as one of its faces. Clearly, such a cube is uniquely determined by $\mathcal{F}$ and p .

Lemma 3.6. Let $\bar{\mathcal{F}} = \{\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1\}$ be the quadrangle of Q with

$$\begin{aligned} \bar{a}^1 &= [1, 0, 0, 0, 0, 0], & \bar{b}^1 &= [0, 0, 1, 0, 0, 0], \\ \bar{c}^1 &= [0, 1, 0, 0, 0, 0], & \bar{d}^1 &= [0, 0, 0, 1, 0, 0]. \end{aligned}$$

Then,

$$I(\bar{\mathcal{F}}) = \left\{ \left[u, \frac{1}{u}, v, \frac{1}{v}, r, s \right] : r^2 + rs + \lambda s^2 = 1, uv \neq 0, u, v \in \mathbb{F}_q \right\} \quad (4)$$

and

$$|I(\bar{\mathcal{F}})| = (q-1)^2(q+1) = (q-1)(q^2-1). \quad (5)$$

Proof. Let $\bar{p} = [p_1, p_2, p_3, p_4, r, s] \in I(\bar{\mathcal{F}})$. Since $\bar{p} \notin Q$, $f(\bar{p}) \neq 0$ (with f as in (2)). Without loss of generality, we can assume that $f(\bar{p}) = 1$, i.e.

$$p_1 p_2 + p_3 p_4 + r^2 + rs + \lambda s^2 = 1. \quad (6)$$

Since $\bar{p}^\perp \cap \bar{\mathcal{F}} = \emptyset$ by definition of $I(\bar{\mathcal{F}})$, $\bar{p}$ is not collinear in Q with any element of $\bar{\mathcal{F}}$. Hence, in terms of coordinates of $\bar{p}$, we have $p_1 p_2 p_3 p_4 \neq 0$. Suppose that $\bar{a}^2, \bar{b}^2, \bar{c}^2, \bar{d}^2$ are points of Q so that $\bar{C} = \{\bar{a}^i, \bar{b}^i, \bar{c}^i, \bar{d}^i\}_{i=1,2}$ is a centric cube of $Q \setminus \bar{p}^\perp$ with center $\bar{p}$. Recall that the exponents 1 and 2 of the same symbol refer to opposite points, that the collinearities between points in $\bar{C}$ are as described in figure 2 and each of $\bar{a}^2, \bar{b}^2, \bar{c}^2$,

$\bar{d}^2$ are on a line joining $\bar{p}$ to $\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1$, respectively. Hence, in terms of coordinates, we have

$$\bar{a}^2 = \bar{a}^1 + t_a \bar{p}, \quad \bar{b}^2 = \bar{b}^1 + t_b \bar{p}, \quad \bar{c}^2 = \bar{c}^1 + t_c \bar{p}, \quad \bar{d}^2 = \bar{d}^1 + t_d \bar{p}$$

for suitable $t_a, t_b, t_c, t_d \in \mathbb{F}_q \setminus \{0\}$. By hypothesis, plugging in the coordinates of $\bar{a}^1, \bar{b}^1, \bar{c}^1, \bar{d}^1$, we have

$$\begin{aligned} \bar{a}^2 &= [1 + t_a p_1, t_a p_2, t_a p_3, t_a p_4, t_a r, t_a s]; \\ \bar{b}^2 &= [t_b p_1, t_b p_2, 1 + t_b p_3, t_b p_4, t_b r, t_b s]; \\ \bar{c}^2 &= [t_c p_1, t_c p_2 + 1, t_c p_3, t_c p_4, t_c r, t_c s]; \\ \bar{d}^2 &= [t_d p_1, t_d p_2, t_d p_3, 1 + t_d p_4, t_d r, t_d s]. \end{aligned}$$

Translating in terms of coordinates the collinearities in Q for $\bar{C}$ to be a cube with center $\bar{p}$, we have

$$\begin{aligned} \bar{a}^1 \perp \bar{c}^2 : t_c p_2 + 1 &= 0; & \bar{b}^1 \perp \bar{d}^2 : t_d p_4 + 1 &= 0; \\ \bar{c}^1 \perp \bar{a}^2 : t_a p_1 + 1 &= 0; & \bar{d}^1 \perp \bar{b}^2 : t_b p_3 + 1 &= 0; \\ \bar{a}^2 \perp \bar{b}^2 : t_b p_2 + t_a p_4 &= 0; & \bar{b}^2 \perp \bar{c}^2 : t_b p_1 + t_c p_4 &= 0; \\ \bar{c}^2 \perp \bar{d}^2 : t_d p_1 + t_c p_3 &= 0; & \bar{d}^2 \perp \bar{a}^2 : t_d p_2 + t_a p_3 &= 0. \end{aligned} \quad (7)$$

Consequently, $p_1 p_2 + p_3 p_4 = 0$ and (6) becomes

$$r^2 + rs + \lambda s^2 = 1. \quad (8)$$

Since $\bar{a}^2, \bar{b}^2, \bar{c}^2, \bar{d}^2$ are points of Q , we have $f(\bar{a}^2) = f(\bar{b}^2) = f(\bar{c}^2) = f(\bar{d}^2) = 0$, i.e., $p_2 = t_a, p_4 = t_b, p_1 = t_c$ and $p_3 = t_d$. The first two rows of (7) give $p_1 p_2 = 1 = p_3 p_4$.

Put $u := p_1$ and $v := p_3$. Then, $\bar{p} = [u, 1/u, v, 1/v, r, s]$ where $uv \neq 0$ and r, s satisfy eq. (8).

By Lemma 3.3 with $\mu = 0$, (8) admits $q + 1$ distinct solutions. So, the lemma follows. $\square$

We will refer to the quadrangle $\bar{\mathcal{F}}$ as in Lemma 3.6 as the *fundamental quadrangle* of Q . By the proof of Lemma 3.6, the coordinates of the points of the cube $C_{\bar{p}}$ having $\bar{\mathcal{F}}$ as a face and $\bar{p} \in I(\bar{\mathcal{F}})$ as the center are the following:

$$\begin{aligned} \bar{a}^1 &= [1, 0, 0, 0, 0, 0], & \bar{b}^1 &= [0, 0, 1, 0, 0, 0], \\ \bar{c}^1 &= [0, 1, 0, 0, 0, 0], & \bar{d}^1 &= [0, 0, 0, 1, 0, 0], \\ \bar{a}^2 &= \left[0, \frac{1}{u^2}, \frac{v}{u}, \frac{1}{uv}, \frac{r}{u}, \frac{s}{u}\right] & \bar{b}^2 &= \left[\frac{u}{v}, \frac{1}{uv}, 0, \frac{1}{v^2}, \frac{r}{v}, \frac{s}{v}\right]; \\ \bar{c}^2 &= \left[u^2, 0, uv, \frac{u}{v}, ur, us\right] & \bar{d}^2 &= \left[uv, \frac{v}{u}, v^2, 0, vr, vs\right] \\ \bar{p} &= \left[u, \frac{1}{u}, v, \frac{1}{v}, r, s\right], \end{aligned} \quad (9)$$

where $u, v \in \mathbb{F}_q$ with $uv \neq 0$ and r, s satisfies (8).

COROLLARY 3.7

Every quadrangle of Q can be extended to $(q - 1)^2(q + 1)$ centric cubes of Q .

Proof. This is straightforward, by eq. (5) of Lemma 3.6 and the transitivity of $SO^-(6, q)$ on the set of quadrangles of Q . $\square$

COROLLARY 3.8

Γ admits non-linear 4-cliques.

Proof. By Theorem 3.5, if a non-linear 4-clique exists, then it can be uniquely lifted to a centric cube with center n_0 . Hence existence of non-linear 4-cliques is equivalent to the existence of centric cubes with center n_0 . By Lemma 3.6, centric cubes having $\tilde{F}$ as a face exist. Let $C_{\tilde{p}}$ be one of these cubes and let $\tilde{p}$ be its center.

The group $SO^-(6, q) \leq \text{Aut}(Q)$ is transitive on the set of points of $PG(5, q) \setminus Q$. Then, there exists $g \in SO^-(6, q)$ such that $g(\tilde{p}) = n_0$. The cube $g(C_{\tilde{p}})$ has the required properties. $\square$

We now prove Theorem 2(c).

Theorem 3.9. Every non-linear 3-clique of Γ can be extended to $q + 1$ non-linear 4-cliques of Γ .

Proof. By Theorems 3.4 and 3.5, each non-linear 3-clique can be extended to $q + 1$ non-linear 4-cliques if and only if each centric hexagon of $\mathcal{X}$ can be extended to $q + 1$ centric cubes of $\mathcal{X}$ having the same center as the hexagon.

Let $H = \{a^i, b^i, c^i\}_{i=1,2}$ be a centric hexagon of Q , with vertices marked as in figure 1.

Since the group $SO^-(6, q)$ is transitive on the set of paths of length 2 of the collinearity graph of Q , we can suppose that a^1, b^1, c^1 are indeed the vertices $\bar{a}^1, \bar{b}^1, \bar{c}^1$ as in the fundamental quadrangle $\tilde{F}$, i.e. $a^1 = [1, 0, 0, 0, 0, 0]$, $b^1 = [0, 0, 1, 0, 0, 0]$ and $c^1 = [0, 1, 0, 0, 0, 0]$.

Let us determine the opposite points a^2, b^2, c^2 . In order to do that, let $p = [p_1, p_2, p_3, p_4, p_5, p_6]$ be the center of H . Hence, p is not collinear in Q with any of a^1, b^1, c^1 . In terms of coordinates, this fact is equivalent to $p_1, p_2, p_3 \neq 0$. Since p is a point of $PG(V)$ not in Q , $f((p_1, p_2, p_3, p_4, p_5, p_6)) \neq 0$. Without loss of generalities, we can suppose that $f((p_1, p_2, p_3, p_4, p_5, p_6)) = 1$, i.e. $p_1 p_2 + p_3 p_4 + p_5^2 + p_5 p_6 + \lambda p_6^2 = 1$. Then there exists $\mu_a, \mu_b, \mu_c \in \mathbb{F}_q \setminus \{0\}$ such that

$$\begin{aligned} a^2 &= [\mu_a(1, 0, 0, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] \\ &= [\mu_a + p_1, p_2, p_3, p_4, p_5, p_6]; \\ b^2 &= [\mu_b(0, 0, 1, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] \\ &= [p_1, p_2, \mu_b + p_3, p_4, p_5, p_6]; \\ c^2 &= [\mu_c(0, 1, 0, 0, 0, 0) + (p_1, p_2, p_3, p_4, p_5, p_6)] \\ &= [p_1, \mu_c + p_2, p_3, p_4, p_5, p_6]. \end{aligned}$$

Since a^2, b^2, c^2 are points of Q and $f(p_1, p_2, p_3, p_4, p_5, p_6) = 1$ by assumption, we get $\mu_a = 1/p_2, \mu_b = 1/p_4, \mu_c = 1/p_1$, so

$$\begin{aligned} a^2 &= [1/p_2 + p_1, p_2, p_3, p_4, p_5, p_6], \\ b^2 &= [p_1, p_2, 1/p_4 + p_3, p_4, p_5, p_6], \\ c^2 &= [p_1, 1/p_1 + p_2, p_3, p_4, p_5, p_6]. \end{aligned} \tag{10}$$

Translating in terms of coordinates the collinearity relations in H (see figure 1), we have

$$\frac{1}{p_2} + p_1 = 0 \text{ because } a^2 \perp c^1 \text{ and } \frac{1}{p_1} + p_2 = 0 \text{ because } c^2 \perp a^1.$$

Hence $p_2 = 1/p_1$. Note that the relations $b^2 \perp a^2$ and $b^1 \perp c^1$ are automatically satisfied. Equations (10) become

$$\begin{aligned} a^2 &= [0, 1/p_1, p_3, p_4, p_5, p_6], \\ b^2 &= [p_1, 1/p_1, 1/p_4 + p_3, p_4, p_5, p_6], \\ c^2 &= [p_1, 0, p_3, p_4, p_5, p_6]. \end{aligned} \quad (11)$$

and $p = [p_1, 1/p_1, p_3, p_4, p_5, p_6]$ with $p_3p_4 + p_5^2 + p_5p_6 + \lambda p_6^2 = 0$.

Let $d^1 = [d_1, d_2, d_3, d_4, d_5, d_6]$ and $d^2 = \mu d^1 + p$ ($\mu \in \mathbb{F}_q \setminus \{0\}$) be two points of Q . If we require the set $H \cup \{d^1, d^2\}$ to be a cube with center p , we obtain the following conditions on the coordinates of d^1 and d^2 : $d_2 = 0$ because $d^1 \perp a^1$ and $d_1 = 0$ because $d^1 \perp c^1$. Also $d_4 \neq 0$ since $d^1 \not\perp b^1$. Hence we can assume $d_4 = 1$. Therefore $d^1 = [0, 0, d_3, 1, d_5, d_6]$. Since $d^1 \perp b^2$, we have

$$d_3p_4 = \frac{1}{p_4} + p_3 + d_5p_6 + d_6p_5. \quad (12)$$

Observe that $d^1 \not\perp b^1$. Indeed

$$\alpha((p_1, 1/p_1, p_3, p_4, p_5, p_6), (0, 0, d_3, 1, d_5, d_6)) = 1/p_4 \neq 0,$$

where α is the bilinear form associated to f . Since d^1 is a point of Q , we get

$$d_3 = d_5^2 + d_5d_6 + \lambda d_6^2. \quad (13)$$

Let us turn to the opposite point $d^2 = \mu d^1 + p$ of d^1 . The fact that d^2 is a point of Q is equivalent to $1 + \mu/p_4 = 0$, hence $\mu = p_4$. So,

$$d^2 = [p_1, 1/p_1, p_4d_3 + p_3, 0, p_4d_5 + p_5, p_4d_6 + p_6].$$

By (12), we get

$$d^2 = [p_1, 1/p_1, p_5d_6 + d_5p_6 + 1/p_4, 0, p_4d_5 + p_5, p_4d_6 + p_6].$$

Note that the relations $d^2 \perp b^1$, $d^2 \perp a^2$, $d^2 \perp c^2$ are automatically satisfied. By (12) and (13) we now have

$$\left(\frac{1}{p_4}\right)^2 + \frac{p_3}{p_4} + d_5\frac{p_6}{p_4} + d_6\frac{p_5}{p_4} + d_5^2 + d_5d_6 + \lambda d_6^2 = 0$$

which is equivalent to

$$\begin{aligned} &\left(d_5 + \frac{p_5}{p_4}\right)^2 + \left(d_5 + \frac{p_5}{p_4}\right)\left(d_6 + \frac{p_6}{p_4}\right) + \lambda\left(d_6 + \frac{p_6}{p_4}\right)^2 \\ &+ \left(\frac{p_5}{p_4}\right)^2 + \frac{p_5p_6}{p_4^2} + \lambda\left(\frac{p_6}{p_4}\right)^2 + \left(\frac{1}{p_4}\right)^2 + \frac{p_3}{p_4} = 0. \end{aligned}$$

Since

$$\begin{aligned} & \left(\frac{p_5}{p_4}\right)^2 + \frac{p_5 p_6}{p_4^2} + \lambda \left(\frac{p_6}{p_4}\right)^2 + \left(\frac{1}{p_4}\right)^2 + \frac{p_3}{p_4} \\ &= \frac{1 + p_3 p_4 + p_5^2 + p_5 p_6 + \lambda p_6^2}{p_4^2} \\ &= \frac{f((p_1, p_2, p_3, p_4, p_5, p_6))}{p_4^2} = \frac{1}{p_4^2} \neq 0, \end{aligned}$$

we can apply Lemma 3.3 with $x := d_5 + \frac{p_5}{p_4}$, $y := d_6 + \frac{p_6}{p_4}$, $\mu = \frac{1}{p_4^2} + 1$ to obtain $q + 1$ different choices for d^1 . The theorem is proved. $\square$

3.3.2 *Non-linear i -cliques*, $i \in \{5, 6\}$. A *decade* of $\hat{\mathcal{X}}$ is a subgraph of $\hat{\Gamma}$ with 10 vertices

$$T = \{a^1, b^1, c^1, d^1, e^1, a^2, b^2, c^2, d^2, e^2\},$$

isomorphic to the complement of the collinearity graph of a (5×2) -grid as shown in figure 5.

The graph T admits a partition of its vertices by two 5-sets (corresponding to the two long lines of the grid) as well as a partition by five 2-sets (corresponding to the five short lines of the grid). Two points in the same 2-set are said to be *opposite*. Clearly, any four (respectively, three) 2-sets of opposite vertices of a decade form a cube (respectively, a hexagon). We say that a decade is *centric* if all the projective lines through opposite vertices are concurrent. The point of concurrence is called the *center of the decade*.

PROPOSITION 3.10

The following hold:

- (a) Every non-linear 5-clique of Γ lifts through φ_0 to a centric decade in $\hat{\Gamma}$ with the nucleus n_0 of Q_0 as the center.
- (b) Every non-linear 6-clique of Γ lifts through φ_0 to a centric dodecade of $\hat{\Gamma}$ with the nucleus n_0 of Q_0 as the center.

Proof. We will only prove (b). The proof of (a) is similar.

Let $\mathcal{C} = \{a, b, c, d, e, f\}$ be a non-linear 6-clique of Γ . Consider the 4-cliques $\{a, b, c, d\}$ and $\{a, f, e, c\}$ of $\mathcal{C}$ and denote their liftings by $\{a^i, b^i, c^i, d^i\}_{i=1,2}$ and $\{a^i, f^i, e^i, c^i\}_{i=1,2}$, respectively. By Theorem 3.5, the lifting of any non-linear 4-subclique of $\mathcal{C}$ is a centric cube of $\hat{\mathcal{X}}$ with center n_0 . Hence, we can lift $\mathcal{C}$ to a subgraph of $\hat{\Gamma}$ which is the complement of figure 6.

Figure 5. (5×2) -grid as complement of a decade.

Clearly, the complement of the graph in figure 6 is a dodecade. By definition of the canonical covering φ_0 , each of the opposite pairs $\{a^1, a^2\}$, $\{b^1, b^2\}$, $\{c^1, c^2\}$, $\{d^1, d^2\}$, $\{e^1, e^2\}$, $\{f^1, f^2\}$ is on a secant line to Q through n_0 . $\square$

Every centric dodecade of $\hat{\Gamma}$ with center n_0 is mapped by φ_0 onto a non-linear 6-clique of Γ . This observation combined with Claim (b) of Proposition 3.10 proves Theorem 3(c). $\square$

The following theorem proves Theorem 2(a), (b).

Theorem 3.11. *The following hold:*

- (a) *The graph Γ admits non-linear 5-cliques if and only if $q = 2^n$, n odd.*
- (b) *If $q = 2^n$, n odd, then every 5-clique of Γ is contained in a 6-clique of Γ .*
- (c) *The graph Γ has no k -cliques with $k > 6$.*

Proof. Let $C := \{a^i, b^i, c^i, d^i\}_{i=1,2}$ be the lifting of a non-linear 4-clique of Γ . By Theorem 3.5, C is a centric cube of $\hat{\mathcal{X}}$. By the transitivity of the action of $SO^-(6, q)$ on quadrangles of Q and the proof of Lemma 3.6, without loss of generality, we can choose the cube $C_{\bar{p}}$ as in (9) of §3.3.1. In particular, $C_{\bar{p}}$ has center $\bar{p} = [u, 1/u, v, 1/v, r, s]$ with $uv \neq 0$ and $r^2 + rs + \lambda s^2 = 1$.

We want to enlarge $C_{\bar{p}}$ to get a lifting of a k -clique of Γ with $k \geq 5$.

Let $\bar{e}^1 = [e_{1,i}]_{i=1}^6$ and $\bar{e}^2 = [e_{2,i}]_{i=1}^6$ be two points of Q . The conditions on the coordinates of $\bar{e}^i$ ($i = 1, 2$) for the set $C_{\bar{p}} \cup \{\bar{e}^i\}_{i=1}^2$ to be a decade of $Q \setminus \bar{p}^\perp$ with center $\bar{p}$ are as follows (see Fig. 6):

$$\begin{aligned} \bar{b}^1 \perp \bar{e}^1 : e_{1,4} &= 0; & \bar{d}^1 \perp \bar{e}^1 : e_{1,3} &= 0; \\ \bar{a}^2 \perp \bar{e}^1 : e_{1,1} &= e_{1,5}us + e_{1,6}ur; & \bar{c}^2 \perp \bar{e}^1 : e_{1,2} &= e_{1,5}s/u + e_{1,6}r/u; \\ \bar{a}^1 \perp \bar{e}^2 : e_{2,2} &= 0; & \bar{c}^1 \perp \bar{e}^2 : e_{2,1} &= 0; \\ \bar{b}^2 \perp \bar{e}^2 : e_{2,3} &= e_{2,5}sv + e_{2,6}rv; & \bar{d}^2 \perp \bar{e}^2 : e_{2,4} &= e_{2,5}s/v + e_{2,6}r/v. \end{aligned} \quad (14)$$

Since $\bar{e}^1$ and $\bar{e}^2$ are points of Q , we have $f(\bar{e}^1) = 0$ and $f(\bar{e}^2) = 0$. Hence

$$e_{1,5}^2(s^2 + 1) + e_{1,5}e_{1,6} + e_{1,6}^2(r^2 + \lambda) = 0, \quad (15)$$

$$e_{2,5}^2(s^2 + 1) + e_{2,5}e_{2,6} + e_{2,6}^2(r^2 + \lambda) = 0, \quad (16)$$

The points $\bar{e}^1$, $\bar{e}^2$ and $\bar{p}$ must be on the same line. Hence

$$e_{2,5} = e_{1,5}(1 + rs) + e_{1,6}r^2 \text{ and } e_{2,6} = e_{1,5}s^2 + e_{1,6}(1 + rs). \quad (17)$$

Figure 6. (6×2) -grid as complement of a dodecade.

By plugging $e_{2,5}$ and $e_{2,6}$ of (17) in (16), we have

$$e_{1,5}^2((r^2 + rs + \lambda s^2)s^2 + 1) + e_{1,5}e_{1,6} + e_{1,6}^2(r^2(r^2 + rs + \lambda s^2) + \lambda) = 0.$$

Since $r^2 + rs + \lambda s^2 = 1$, the above equation gives us back (15). So, (16) is already implicit in (15). Hence we can disregard it.

The coordinates $e_{1,i}$ of $\bar{e}^1$ for $1 \leq i \leq 4$ and the coordinates $e_{2,i}$ of $\bar{e}^2$ for $1 \leq i \leq 4$ depend on $e_{1,5}$ and $e_{1,6}$. Hence we write $\bar{e}_{(e_{1,5}, e_{1,6})}^1$ for $\bar{e}^1$ and $\bar{e}_{(e_{1,5}, e_{1,6})}^2$ for $\bar{e}^2$ to emphasize the dependance of $\bar{e}^1$ and $\bar{e}^2$ on $e_{1,5}$ and $e_{1,6}$. Explicitly, we have

$$\bar{e}_{(e_{1,5}, e_{1,6})}^1 = (e_{1,5}us + e_{1,6}ur, e_{1,5}\frac{s}{u} + e_{1,6}\frac{r}{u}, 0, 0, e_{1,5}, e_{1,6}) \quad (18)$$

and

$$\begin{aligned} \bar{e}_{(e_{1,5}, e_{1,6})}^2 = & (0, 0, e_{1,5}sv + e_{1,6}rv, \frac{se_{1,5}}{v} + \frac{re_{1,6}}{v}, \\ & e_{1,5}(1 + rs) + e_{1,6}r^2, e_{1,5}s^2 + e_{1,6}(1 + rs)). \end{aligned} \quad (19)$$

Note that the possibility to enlarge $C_{\bar{p}}$ in order to get a lifting of a non-linear k -clique with $k \geq 5$ is equivalent to the existence of at least two points $\bar{e}^1$ and $\bar{e}^2$ as above. The number of choices for the pair $(\bar{e}^1, \bar{e}^2)$ determines the size of the k -clique. After having implemented all the collinearity relations among the points $C_{\bar{p}} \cup \{\bar{e}^1, \bar{e}^2\}$ to get a centric decade, we get that the points $\bar{e}^1$ and $\bar{e}^2$ exist provided their coordinates satisfy (8) and (15). In the sequel, we will show that (8) and (15) are equivalent to $\text{Tr}(1) = 1$. Then, $\text{Tr}(1) = 1$ if and only if q is an odd power of 2 (see e.g. [8]).

More precisely, we will deal with the cases $s = 1$ and $s \neq 1$ separately showing that

- (i) for $s = 1$ and $s \neq 1$ the choices for $(\bar{e}^1, \bar{e}^2)$ allow exactly two extensions of $C_{\bar{p}}$ to decades;
- (ii) the union of the two decades in (i) is a centric dodecade.

Suppose firstly $s = 1$. Then (15) is

$$e_{1,5}e_{1,6} + e_{1,6}^2(r^2 + \lambda) = 0 \quad (20)$$

and (8) becomes $\lambda = r^2 + r + 1$. So, $\text{Tr}(\lambda) = \text{Tr}(r^2 + r + 1) = \text{Tr}(1)$. Since $\text{Tr}(\lambda) = 1$, we get

$$\text{Tr}(1) = 1. \quad (21)$$

Suppose now $s \neq 1$. Then (15) is

$$e_{1,5}^2 + \frac{e_{1,5}e_{1,6}}{1 + s^2} + \frac{(\lambda + r^2)(1 + s^2)e_{1,6}^2}{(1 + s^2)^2} = 0. \quad (22)$$

Equation (22) admits solutions (in $e_{1,5}$ and $e_{1,6}$) if and only if

$$\text{Tr}(\lambda + \lambda s^2 + r^2 + r^2 s^2) = 0. \quad (23)$$

By (8) we have $\lambda s^2 + r^2 + r^2 s^2 = 1 + rs + r^2 s^2$. Hence (23) becomes $\text{Tr}(\lambda + \lambda s^2 + r^2 + r^2 s^2) = \text{Tr}(\lambda + 1 + rs + r^2 s^2) = 0$ which implies that $\text{Tr}(\lambda) + \text{Tr}(1) + \text{Tr}(rs) + \text{Tr}(r^2 s^2) = 0$, i.e. $\text{Tr}(1) = 1$, again.

Points $\bar{e}^1$ and $\bar{e}^2$ exist if and only if (21) is satisfied, which is equivalent to require $q = 2^n$ with n odd. So, non-linear 5-cliques exist if and only if $q = 2^n$ with n odd. Claim (a) is proved.

Let $q = 2^n$ and n odd. Then (20) and (22) admit exactly two different solutions (in $e_{1,5}$ and $e_{1,6}$), up to proportionality: denote them by $(e_{1,5}^*, e_{1,6}^*)$ and $(e_{1,5}^\circ, e_{1,6}^\circ)$. According to (18) and (19) and keeping in mind that the exponents 1 and 2 refer to opposite points, put

$$\bar{e}^{*1} := \bar{e}_{(e_{1,5}^*, e_{1,6}^*)}^1, \quad \bar{e}^{*2} := \bar{e}_{(e_{1,5}^*, e_{1,6}^*)}^2, \quad \bar{e}^{\circ 1} := \bar{e}_{(e_{1,5}^\circ, e_{1,6}^\circ)}^1, \quad \bar{e}^{\circ 2} := \bar{e}_{(e_{1,5}^\circ, e_{1,6}^\circ)}^2.$$

By construction, $C_{\bar{p}} \cup \{\bar{e}^{*1}, \bar{e}^{*2}\}$ and $C_{\bar{p}} \cup \{\bar{e}^{\circ 1}, \bar{e}^{\circ 2}\}$ are centric decades with center $\bar{p}$ and $C_{\bar{p}} \cup \{\bar{e}^{*1}, \bar{e}^{*2}, \bar{e}^{\circ 1}, \bar{e}^{\circ 2}\}$ is a centric dodecade with center $\bar{p}$. The theorem is proved. $\square$

The following is also implicit in the proof of Theorem 3.11 and corresponds to Theorem 2(d)(i) and (ii).

COROLLARY 3.12

Let n be odd. Then every centric cube of $\hat{\mathcal{X}}$ is contained in exactly two centric decades and just one centric dodecade of $\hat{\mathcal{X}}$. Every centric decade of $\hat{\mathcal{X}}$ is contained in exactly one centric dodecade of $\hat{\mathcal{X}}$. $\square$

PROPOSITION 3.13

Let $\mathcal{N}_i, i \in \{2, 3, 4, 5, 6\}$, be the set of non-linear i -cliques in Γ and $|\mathcal{N}_i| = N_i$. Then

- (a) $N_2 = q^2(q-1)(q^4-1)/4$; $N_3 = q^4(q-1)(q^4-1)/12$; $N_4 = q^4(q^2-1)(q^4-1)/48$.
- (b) *Let $q = 2^n > 4$. If n is even, then $\mathcal{N}_5 = \mathcal{N}_6 = \emptyset$. If n is odd, then $N_5 = q^4(q^2-1)(q^4-1)/120$ and $N_6 = q^4(q^2-1)(q^4-1)/720$.*

Proof. Each non-linear i -clique, $i \in \{2, 3, 4, 5\}$ in Γ extends to a non-linear $(i+1)$ -clique in Γ in q^2 ways if $i = 2$ (see calculation of λ in §3.1); in $q+1$ ways if $i = 3$ (Theorem 3.9); in two ways if n is odd and $i = 4$; no way if n even and $i = 4$; and in a unique way if n is odd and $i = 5$ (Corollary 3.12). Any non-linear $(i+1)$ -clique can be considered as an extension of any of its i -subcliques. Further, the number N_2 of 2-cliques (i.e., edges) in Γ is $vk/2$. So, $N_3 = q^2 N_2/3$, $N_4 = (q+1)N_3/4$, $N_5 = N_6 = 0$ if n is even and, if n is odd, $N_5 = 2N_4/5$ and $N_6 = N_5/6$. So, the proposition follows. $\square$

3.4 Proof of Theorem 4

(a) Let $H = \{a^i, b^i, c^i\}_{i=1,2}$ be a centric hexagon of $\hat{\mathcal{X}}$ with center n_0 . The lines $\langle a^1, b^1 \rangle$ and $\langle a^2, b^2 \rangle$ are coplanar. So, they meet at a point $a^0 \in H_0 \cap Q$, where $H_0 := n_0^\perp$. Similarly, the lines $\langle b^1, c^1 \rangle$ and $\langle b^2, c^2 \rangle$ meet at a point $b^0 \in H_0 \cap Q$ and the lines $\langle a^1, c^2 \rangle$ and $\langle a^2, c^1 \rangle$ meet at the point $c^0 \in H_0 \cap Q$. Let $S := \{a^i, b^i, c^i\}_{i=0,1,2}$ and consider the subgeometry $Q(S)$ of Q induced on S (see §1.1, paragraph before Theorem 4, for the definition), namely $Q(S) := (S, \mathcal{L}_S)$ where $\mathcal{L}_S = \{l \cap S : l \text{ is a line of } Q \text{ such that } |l \cap S| \geq 2\}$.

It is straightforward to see that the subgeometry $Q(S)$ is an $\mathbb{F}_2$ -subgeometry isomorphic to a classical $(2, 1)$ -generalized quadrangle $Q^+(3, 2)$. By construction, the points a^0, b^0, c^0 are orthogonal to n_0 (with respect to the orthogonality relation $\perp$ defined by Q). Hence $H \cong Q^+(3, 2) \setminus n_0^\perp$.

(b) Let $C = \{a^i, b^i, c^i, d^i\}_{i=1,2}$ be a cube of $\hat{\mathcal{X}}$ with center n_0 and $\{\hat{l}_1, \hat{m}_1\}, \{\hat{l}_2, \hat{m}_2\}, \dots, \{\hat{l}_6, \hat{m}_6\}$ be the six pairs of opposite edges of C . Then, for $j = 1, \dots, 6$, the lines $\hat{l}_j$ and $\hat{m}_j$ have the same point at infinity $l_j^\infty = m_j^\infty =: p_j$. These six points $p_1, p_2, \dots, p_6$ are partitioned into three pairs and each of these pairs is collinear in Q_0 with a given point $p_0 \in Q_0$. Each of these three pairs of points (on a same line through p_0) corresponds to one of the three classes of parallel edges of the cube. The point p_0 is obtained as the sum of the 4 vertices of any face of the cube.

Let $S := \{a^i, b^i, c^i, d^i\}_{i=1,2} \cup \{p_i\}_{0 \leq i \leq 6}$ and consider the subgeometry $Q(S)$ of Q induced on S .

It is straightforward to see that $Q(S)$ is an $\mathbb{F}_2$ -subgeometry isomorphic to $Q(4, 2)$. It is also well known that the complement in $Q(4, 2)$ of a tangent hyperplane is a cube. Hence, $C \cong Q(S) \setminus p_0^\perp$. Note that the projective subgeometry $\langle S \rangle_{\mathbb{F}_2}$ defined by S over the field $\mathbb{F}_2$ contains the nucleus n_0 of Q_0 but n_0 is not the nucleus n_S of the quadric $Q(S)$ (embedded in $\langle S \rangle_{\mathbb{F}_2}$).

(c) As remarked in §1, if $q = 2$ then Γ is a complete graph with 6 vertices. In this case, $Q \setminus Q_0$ is a dodecade.

Let $q \geq 4$. Since by assumption q is an odd power of 2, we can take $\lambda = 1$ in (2); thus Q is represented by the following equation:

$$x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + x_6^2 = 0. \quad (24)$$

We shall count the number of subgeometries of Q defined over the field $\mathbb{F}_2$ and represented by eq. (24) for a suitable choice of a basis. Equivalently, we count the number of different bases B_i of the vector space $V = V(6, q)$ with the property that with respect to each of these bases B_i , the quadratic form induced by f on the $\mathbb{F}_2$ -space spanned by B_i is a quadratic form of elliptic type over $\mathbb{F}_2$, hence equivalent to $x_1x_2 + x_3x_4 + x_5^2 + x_5x_6 + x_6^2$.

Let us construct a basis $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ as required above. We can choose b_1, b_2, b_3, b_4 so that $\langle b_1 \rangle, \langle b_2 \rangle, \langle b_3 \rangle, \langle b_4 \rangle$ is a quadrangle of Q . That is, $f(b_i) = 0$ for $1 \leq i \leq 4$, $b_1 \not\perp b_3$, $b_2 \not\perp b_4$ and $b_1 \perp b_2 \perp b_3 \perp b_4 \perp b_1$.

The number of quadrangles of Q is $(q^3 + 1)(q^2 + 1)(q + 1)q^6$.

The condition $\alpha(b_1, b_2) = 1$ and $\alpha(b_3, b_4) = 1$ (where α stands for the bilinear form associated to f) still allows us to choose b_1 (or b_2) and b_3 (or b_4) up to arbitrary scalars. Hence, we have $(q^3 + 1)(q^2 + 1)(q + 1)q^6(q - 1)^2$ choices for the quadruple b_1, b_2, b_3, b_4 . The vectors b_5 and b_6 are taken in the orthogonal space $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ of $\langle b_1, b_2, b_3, b_4 \rangle$. (Recall that $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ is a projective line exterior to Q .) So, let us regard b_5 and b_6 as vectors of the 2-dimensional vector space $\langle b_1, b_2, b_3, b_4 \rangle^\perp$.

Let $f'(x, y) = x^2 + \mu xy + \nu y^2$ be the quadratic form induced by f on $\langle b_1, b_2, b_3, b_4 \rangle^\perp$, for a suitable choice of a basis e_1, e_2 of $\langle b_1, b_2, b_3, b_4 \rangle^\perp$. Hence $\text{Tr}(\nu/\mu^2) = 1$ because the line $\langle b_1, b_2, b_3, b_4 \rangle^\perp$ is external to Q . Put $b_5 := x_1e_1 + x_2e_2$ and $b_6 := y_1e_1 + y_2e_2$ for scalars x_1, x_2, y_1, y_2 to be determined in a few lines.

The conditions that are required on b_5 and b_6 are the following: $\alpha(b_5, b_6) = 1$, $f(b_5) = 1$ and $f(b_6) = 1$. By Lemma 3.3, we have $q + 1$ different choices for b_5 . Indeed, $f(b_5) = 1$ can be translated to $x_1^2 + x_1x_2 + x_2^2 = 1$.

Having chosen b_5 , let us count how many choices remain for b_6 .

To fix ideas, put $b_5 = e_1$. Then $\alpha(b_5, b_6) = 1$ implies $y_2 = 1/\mu$ and $f(b_6) = 1$ implies $y_1^2 + \mu y_1 y_2 + \nu y_2^2 = 1$, namely $y_1^2 + y_1 + \nu/(\mu^2) + 1 = 0$. Since the equation $y_1^2 + y_1 + \nu/(\mu^2) + 1 = 0$ has two distinct solutions, two different choices are left for b_6 . A similar argument works with any other choice of b_5 . Thus, in any case, only two choices are left for b_6 once b_5 has been chosen. So, the number of $\mathbb{F}_2$ -subgeometries $Q(S)$ of $Q^-(5, q)$ isomorphic to $Q^-(5, 2)$ is

$$\begin{aligned} \frac{\# \text{ choices of } (b_i)_{i=1}^6 \text{ in } V(6, q)}{\# \text{ choices of } (b_i)_{i=1}^6 \text{ in } V(6, 2)} &= \frac{(q^3+1)(q^2+1)(q+1)q^6(q-1)^2(q+1)2}{(2^3+1)(2^2+1)(2+1)2^6(2-1)^2(2+1)2} \\ &= \frac{(q^3+1)(q^2+1)(q+1)^2q^6(q-1)^2}{9 \cdot 5 \cdot 3^2 \cdot 2^6}. \end{aligned} \quad (25)$$

By Proposition 3.13, the total number $\bar{N}_6$ of centric dodecades of Q (any possible center being allowed) is

$$\begin{aligned} &(\# \text{ of dodecades with a given center}) \cdot |\text{PG}(V) \setminus Q| \\ &= N_6 \cdot |\text{PG}(V) \setminus Q| = \frac{(q^4-1)(q^2-1)q^4}{15 \cdot 48} \cdot \left[\frac{(q^6-1)}{q-1} - (q^3+1)(q+1) \right] \\ &= \frac{(q^3+1)(q^4-1)(q^2-1)q^6}{48 \cdot 15}. \end{aligned}$$

$$\text{Hence } \bar{N}_6 = \frac{(q^3+1)(q^4-1)(q^2-1)q^6}{48 \cdot 15}.$$

Note that if $Q(S)$ is an $\mathbb{F}_2$ -subgeometry of Q and $p \in \langle S \rangle_{\mathbb{F}_2} \setminus S$, then $Q(S) \setminus p^\perp$ is a dodecade with center p . For each such subgeometry $Q(S)$, there are $2^2(2^3+1) = 4 \cdot 9$ possible choices for p . In view of (25), this accounts for a total of

$$\frac{(q^3+1)(q^2+1)(q+1)^2q^6(q-1)^2}{9 \cdot 5 \cdot 3^2 \cdot 2^6} \cdot 4 \cdot 9 = \frac{(q^3+1)(q^4-1)(q^2-1)q^6}{5 \cdot 9 \cdot 2^4} = \bar{N}_6$$

centric dodecades. Hence every dodecade with center p corresponds to a subgeometry $Q(S) \setminus p^\perp$ over $\mathbb{F}_2$ as described above and vice versa. $\square$

Acknowledgement

The authors wish to thank Antonio Pasini for his very helpful remarks and comments on a first version of this paper.

References

- [1] Bagchi B and Sastry N S N, One-step completely orthogonalizable codes from generalized quadrangles, *Inf. Comput.* **77** (1988) 123–130
- [2] Brouns L, Thas J A and Van Maldeghem H, A characterization of $Q(5, q)$ using one subquadrangle $Q(4, q)$, *Europ. J. Comb.* **23** (2002) 163–177
- [3] De Bruyn B and Pasini A, Minimal underlying division rings of sets of points of a projective space, *J. Algebra* **318** (2007) 641–652
- [4] De Clerck F and Van Maldeghem H, Some classes of rank 2 geometries, in: Chapter 10 of handbook of incidence geometry (ed.) F Buekenhout (1995) (Elsevier)

- [5] Debroey I and Thas J A, On semipartial geometries, *J. Combin. Theory Ser. A* **25(3)** (1978) 242–250
- [6] Hirshfeld J W P and Thas J A, General Galois geometries (1991) (Oxford: Clarendon Press)
- [7] Hubaut X and Metz R, A class of strongly regular graphs related to orthogonal groups (1983), *Combinatorics '81* (Rome, 1981), *Ann. Discrete Math.*, vol. 18 North-Holland, Amsterdam-New York, pp. 469–472
- [8] Lidl R and Neiderreiter H, Finite fields, in: *Encyclopedia of mathematics and its applications* (1983) (Addison-Wesley Publishing Company), vol. 20
- [9] Pasini A, *Diagram geometries* (1994) (Oxford: Oxford University Press)
- [10] Payne S E and Thas J A, *Finite generalized quadrangles*, 2nd ed. (2009) (Zürich: EMS Series of Lectures in Mathematics)
- [11] Pralle H, Affine generalized quadrangles- an axiomatization, *Geometriae Dedicata* **84** (2001) 1–23
- [12] Thas J A and Payne S E, Spreads and ovoids in finite generalized quadrangles, *Geometriae Dedicata* **52** (1994) 227–253

COMMUNICATING EDITOR: Parameswaran Sankaran