What do analytic functions look like?

Tirthankar Bhattacharyya
Indian Institute of Science, Bangalore

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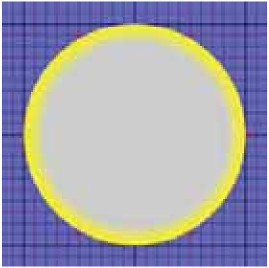

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where a is in $\mathbb{D}$.

A motivating example

The function

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Consider the matrix $\begin{pmatrix} -a & (1-|a|^2)^{1/2} \\ (1-|a|^2)^{1/2} & \bar{a} \end{pmatrix}$

It can be checked that this matrix is unitary.

A very elementary result

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We become more ambitious.

Enter Hilbert spaces

The n-dimensional Euclidean space is

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_i \in \mathbb{C}\}$$

equipped with its standard inner product

$$\langle (z_1, z_2, \ldots, z_n), (w_1, w_2, \ldots, w_n) \rangle = z_1\overline{w_1} + \cdots z_n\overline{w_n}.$$

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A Hilbert space is

$$H = \{(z_1, z_2, \ldots) : |z_1|^2 + |z_2|^2 + \cdots < \infty\}$$

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A bounded operator between two Hilbert spaces is a (possibly infinite) matrix. For example, an $m \times n$ matrix is such an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$.

The formula for the disc

Let there be a Hilbert space H and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$$

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A is a complex number,

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Consider the function $f(z) = A + zB(I - zD)^{-1}C$. The way it is concocted, it is an analytic function on $\mathbb{D}$.

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The formula for the disc continued...

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The surprise is that all analytic functions on the unit disc whose moduli do not exceed 1 are of this form: a function f is in $H^\infty(\mathbb{D})$ and satisfies $|f(z)| \le 1$ for all $z \in \mathbb{D}$ if and only if there is a Hilbert space H and a contraction (iff an isometry iff a unitary operator)

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$$

such that $f(z) = A + zB(I - zD)^{-1}C$.

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There is a branch called H^∞-control where this has been of great use.

Bidisk

The bidisk is the set $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$. It is a subset of $\mathbb{C}^2$. Thus a typical element is $z = (z_1, z_2)$ with $|z_1|, |z_2| < 1$.

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It was shown by Agler that a function f is analytic on $\mathbb{D}^2$ and satisfies $|f(z)| \leq 1$ for all z in the bidisk if and only if there is a graded Hilbert space $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ and a unitary operator $V : \mathbb{C} \oplus \mathcal{L} \to \mathbb{C} \oplus \mathcal{L}$ which can be written in the block form as

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that writing P_1 for projection of $\mathcal{L}$ onto $\mathcal{L}_1$ and P_2 for projection of $\mathcal{L}$ onto $\mathcal{L}_2$,

$$f(z_1, z_2) = A + B(z_1 P_2 + z_2 P_2)[I - (z_1 P_1 + z_2 P_2)D]^{-1} C.$$

The realization formula can be written as

$$f(z_1, z_2) = A + B \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} D \right)^{-1} C$$

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Understanding the bidisk phenomenon better

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The co-ordinate functions z_1 and z_2 are the two functions that determine the bidisk.

What happens when there are more co-ordinate functions?

Take the example of the symmetrized bidisk:

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : z_1 \text{ and } z_2 \text{ are from the open unit disk}\}.$$

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We need to introduce a parametrized family of functions. For $\alpha \in \overline{\mathbb{D}}$ and $(s, p) \in \mathbb{G}$, let

$$\varphi(\alpha, s, p) = \frac{2\alpha p - s}{2 - \alpha s}$$

which is defined for all (α, s, p) satisfying $2 - \alpha s \neq 0$.

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Since $|s| < 2$ for all $(s, p) \in \mathbb{G}$, this function is well-defined on $\overline{\mathbb{D}} \times \mathbb{G}$.

The parametrized co-ordinate functions

The notation $\varphi(\alpha, \cdot)$ will mean that for a fixed α, we are considering it as a function on $\mathbb{G}$ and $\varphi(\cdot, s, p)$ will mean that for a fixed (s, p), we are considering it as a function on $\overline{\mathbb{D}}$.

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These functions are called the parametrized co-ordinate functions because Agler and Young proved that

$$(s, p) \in \mathbb{G} \text{ if and only if } \varphi(\alpha, s, p) \in \mathbb{D} \tag{1}$$

for all α in the closed unit disk.

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For every $\alpha \in \overline{\mathbb{D}}$, the function $\varphi(\alpha, \cdot)$ is an analytic function in $\mathbb{G}$ with modulus at any point not exceeding 1 and for every $(s, p) \in \mathbb{G}$, the function $\varphi(\cdot, s, p)$ is a continuous function on the closed unit disk.

Realization theorem. *The following are equivalent.*

H *f is a function in $H^\infty(\mathbb{G})$ with $\|f\|_\infty \le 1$.*

M $(1 - f(s,p)\overline{f(t,q)})k((s,p),(t,q))$ *is a weak kernel for every admissible kernel k.*

D *There is a positive semi definite kernel $\Delta : \mathbb{G} \times \mathbb{G} \to C(\overline{\mathbb{D}})^*$ such that*

$$1 - f(s,p)\overline{f(t,q)} = \Delta((s,p),(t,q))\big(1 - \varphi(\cdot,s,p)\overline{\varphi(\cdot,t,q)}\big).$$

R *There is a Hilbert space H, a unital $*$-representation $\pi : C(\overline{\mathbb{D}}) \to B(H)$ and a unitray $V : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$ such that writing V as*

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have
$f(s,p) = A + B\pi(\varphi(\cdot,s,p))\big(I_H - D\pi(\varphi(\cdot,s,p))\big)^{-1}C.$

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Russell, Bertrand (1919). "The Study of Mathematics". Mysticism and Logic: And Other Essays. Longman. p. 60.

Thank you for your attention.