THEOREM 2.

For n greater than 1,

$$\sum_{a_1=1}^{(p_1-1)} \cdots \sum_{a_n=1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^{J^2} = 2X + 2g(X) \tag{6}$$

where

$$g(X) = \sum a_m X^{m^2}$$

$$1 < m \leqslant [K] Q \prod_{i=1}^{n} (1-(1/p_i)), \ (m, Q) = 1$$

and a_m as defined in Theorem 1.

THEOREM 3.

For n greater than 1,

Let $(a'_1, a'_2, a'_3, \ldots, a'_n)$ be the unique solution of $-2 \equiv \Sigma_1^n a_i Q_i \bmod Q$, with $0 \leqslant a_i \leqslant (p_i - 1)$, $(1, 2, 3, \ldots, n)$.

Then,

$$\sum_{a'_1 \neq a_1 = 1}^{(p_1-1)} \cdots \sum_{a'_n \neq a_n = 1}^{(p_n-1)} \sum_{0 \leqslant b \leqslant [K]} X^J = \sum_{(m(m+2), Q) = 1} X^m \tag{7}$$

where m on the right side performs summation over the relevant range.

THEOREM 4.

For n greater than 1,

$$\sum_{a_1} \cdots \sum_{a_n} \sum_{b} X^{J^2} = \sum_{(m(m+2), Q) = 1} X^{m^2} \tag{8}$$

where m performs summation over the same set of integers as in Theorem 3.

Remark 1. Note that $a'_1 = 0$ and that $1 \leqslant a'_1 \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$.

Remark 2. In all the four theorems, the m's which satisfy

$$(p_n + 1) < m < (p_{n+1}^2 - 1) \tag{9}$$

are precisely all the primes in this interval.

2. Proofs

The proofs of the theorems 1 and 2 follow from the remarks given below.

First, given any integer c there is a unique solution of

$$\sum_{i=1}^{n} a_i Q_i \equiv c \bmod Q \tag{10}$$

subject to $0 \leqslant a_i \leqslant (p_i - 1)$, $(i = 1, 2, 3, \ldots, n)$. The proofs of the theorems 3 and 4 follow from the remarks given below.